Transfer Matrices for the Zero-Temperature Potts Antiferromagnet on Cyclic and Möbius Lattice Strips

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We present transfer matrices for the zero-temperature partition function of the q-state Potts antiferromagnet (equivalently, the chromatic polynomial) on cyclic and Möbius strips of the square, triangular, and honeycomb lattices of width L_y and arbitrarily great length L_x. We relate these results to our earlier exact solutions for square-lattice strips with $L_y = 3, 4, 5$, triangular-lattice strips with $L_y = 2, 3, 4$, and honeycomb-lattice strips with $L_y = 2, 3$ and periodic or twisted periodic boundary conditions. We give a general expression for the chromatic polynomial of a Möbius strip of a lattice Λ and exact results for a subset of honeycomb-lattice transfer matrices, both of which are valid for arbitrary strip width L_y. New results are presented for the $L_y = 5$ strip of the triangular lattice and the $L_y = 4$ and $L_y = 5$ strips of the honeycomb lattice. Using these results and taking the infinite-length limit $L_x \to \infty$, we determine the continuous accumulation locus of the zeros of the above partition function in the complex q plane, including the maximal real point of nonanalyticity of the degeneracy per site, W as a function of q.

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I. INTRODUCTION

The q-state Potts antiferromagnet (AF) [1,2] exhibits nonzero ground state entropy, $S_0 > 0$ (without frustration) for sufficiently large q on a given lattice Λ or, more generally, on a graph G. This is equivalent to a ground state degeneracy per site $W > 1$, since $S_0 = k_B \ln W$. There is a close connection with graph theory here, since the zero-temperature partition function of the above-mentioned q-state Potts antiferromagnet on a graph $G = G(V, E)$ defined by vertex and edge sets V and E satisfies

$$Z(G, q, T = 0)_{PAF} = P(G, q) \tag{1.1}$$

where $P(G, q)$ is the chromatic polynomial expressing the number of ways of coloring the vertices of the graph G with q colors such that no two adjacent vertices have the same color [3]- [6]. Thus

$$W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n} \tag{1.2}$$

where $n = |V|$ denotes the number of vertices of the graph G and the symbol $\{G\}$ formally denotes the set $\lim_{n \to \infty} G$. The minimum number of colors that is necessary to color a graph G subject to this constraint is called the chromatic number of G, $\chi(G)$.

We represent a given strip as extending longitudinally in the x direction and transversely in the y direction, with width L_y vertices. Each strip involves a longitudinal repetition of m copies of a particular subgraph. For the square-lattice strips, this is a column of squares. It is convenient to represent the strip of the triangular lattice as obtained from the corresponding strip of the square lattice with additional diagonal edges connecting, say, the upper-left to lower-right vertices in each square. In both these cases, the length is $L_x = m$ vertices. We represent the strip of the honeycomb lattice in the form of bricks oriented horizontally. In this case, since there are two vertices in 1-1 correspondence with each horizontal side of a brick, $L_x = 2m$ vertices.

The general structure of the Potts model partition function for cyclic strips of the square lattice of width L_y, as a sum of powers of eigenvalues of a formal transfer matrix multiplied by certain coefficients $c^{(d)}$, $0 \le d \le L_y$, was given in Refs. [7] (see also [8]). The present authors (unaware of this finding in [7]) rediscovered the result in Ref. [9] and showed that it applies also to cyclic strips of the triangular and honeycomb lattices [10]. The coefficients are polynomials of degree d in the variable q given by [7,9]

$$c^{(d)} = U_{2d}(q^{1/2}/2) = \sum_{j=0}^{d} (-1)^j \binom{2d - j}{j} q^{d-j} \tag{1.3}$$

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with $U_n(x)$ being the Chebyshev polynomial of the second kind. The first few of these coefficients are $c^{(0)} = 1$, $c^{(1)} = q - 1$, $c^{(2)} = q^2 - 3q + 1$, and $c^{(3)} = q^3 - 5q^2 + 6q - 1$. The $c^{(d)}$'s play a role analogous to multiplicities of eigenvalues $\lambda_{\Lambda, L_y, d, j}$, although this identification is formal, since $c^{(d)}$ may be zero or negative for the physical values $q = 1, 2, 3$. For example, as shown in eqs. (2.19), (2.20) of Ref. [9], if $q = 2$, then $c^{(d)} = -1$ for $d = 2$ or 3 mod 4, and if $q = 3$, then $c^{(d)} = -1$ for $d = 3$ or 5 mod 6 and $c^{(d)} = -2$ for $d = 4$ mod 6. In general, $c^{(d)}$ vanishes at $q = q_{d,k}$, where $q_{d,k} = 4\cos^2(\pi k / (2d + 1))$, $k = 1, 2, ..., d$, so that $c^{(d)}$ is positive for $q \geq 4$ for arbitrary d.

This structure also holds for the chromatic polynomial, with the difference that, for $L_y \geq 2$, the number of eigenvalues for each d is smaller than the number for the full partition function. We determined this number for strips of the square and triangular lattice in Ref. [9], for strips of the honeycomb lattice in Ref. [10], and for self-dual strips of the square lattice in Ref. [11] (where we also noted how these numbers fill out the entries in the relevant Bratelli diagrams). The chromatic polynomial for a cyclic strip of the regular lattice Λ has the form

$$P(\Lambda, L_y \times m, cyc., q) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_P(\Lambda, L_y, d)} (\lambda_{\Lambda, L_y, d, j})^m \qquad (1.4)$$

Let $G' = (V, E')$ be a spanning subgraph of G, i.e. a subgraph having the same vertex set V and an edge set $E' \subseteq E$. Then $P(G, q)$ can be written as the sum [12,13]

$$P(G, q) = \sum_{G' \subseteq G} q^{k(G')} (-1)^{|E'|} \qquad (1.5)$$

where $k(G')$ denotes the number of connected components of G' and $|E'|$ denotes the number of edges in the set E'. Since we only consider connected graphs G, we have $k(G) = 1$. The formula (1.5) enables one to generalize q from $\mathbb{Z}_+$ to $\mathbb{R}_+$. The zeros of $P(G, q)$ in the complex q plane are called chromatic zeros. We denote the continuous accumulation set of these zeros in the $n \to \infty$ limit as $\mathcal{B}$, which is the continuous locus of points where $W(\{G\}, q)$ is nonanalytic. The maximal value of q where $\mathcal{B}$ intersects the (positive) real axis is labelled $q_c(\{G\})$. This locus occurs as a solution to the degeneracy in magnitude of λ's of maximal magnitude [14]- [17].

In this paper we present transfer matrices T_{Λ, L_y} for strips of the square (sq), triangular (tri), and honeycomb (hc) lattices of fixed transverse widths L_y and arbitrarily great length with periodic (cyclic) longitudinal boundary conditions. We also give results for the corresponding strips with twisted periodic (Möbius) longitudinal boundary conditions. We relate these results to our earlier exact solutions for square-lattice strips with $L_y = 3, 4, 5$;

triangular-lattice strips with $L_y = 2, 3, 4$, and honeycomb-lattice strips with $L_y = 2, 3$ and periodic or twisted periodic boundary conditions. We give a general expression for the chromatic polynomial of a Möbius strip of a lattice Λ which is valid for arbitrary strip width L_y. New results are presented for the $L_y = 5$ strip of the triangular lattice and the $L_y = 4$ and $L_y = 5$ strips of the honeycomb lattice. Using these results and taking the infinite-length limit $L_x \to \infty$, we determine certain properties of the continuous accumulation locus of the zeros of the above partition function in the complex q plane, including the maximal point of nonanalyticity of the degeneracy per site, W as a function of q. We also give a general trace formula for the honeycomb strip which is valid for arbitrary L_y. Chromatic numbers for these lattice strips are given in the appendix.

There are several motivations for this work. One is that, as noted above, the Potts antiferromagnet has the interesting property of nonzero ground state entropy for sufficiently large q, which is an exception to the third law of thermodynamics [18,19]. Via eq. (1.2), exact calculations of $P(G, q)$ yield calculations of $W\{G\}, q)$ and thus give insight into this property. The constraint that no two adjacent vertices have the same value of q becomes less and less important as $q \to \infty$, and in this limit, $W(\{G\}, q) \to q$. As q decreases, $W(\{G\}, q)$ remains a real analytic function of q down to the point $q_c(\{G\})$, where it is nonanalytic. In earlier work, we studied how $q_c(\{G\})$ depends on the boundary conditions used for various lattice strips [20]- [39]. It was found that a convenient property of strips with periodic (or twisted periodic) longitudinal boundary conditions was that they always defined a value of $q_c(\{G\})$, and, furthermore, for a given type of lattice, this point was observed to be a monotonically nondecreasing function of the strip width for all cases calculated. By carrying out exact calculations of $P(G, q)$ and determining the analytic structure of $W(\{G\}, q)$, one can study how $q_c(\{G\})$ varies with increasing strip width and how it approaches the values expected for the corresponding two-dimensional lattices, namely $q_c(sq) = 3$ [40] and $q_c(tri) = 4$ [41,42]. This interpolation property is an especially interesting use of these exact results since the resultant values of $q_c(\{G\})$ interpolate between the value $q_c = 2$ for one-dimension and values on two-dimensional lattices. Similar results were found for the ground state entropy itself [20,44–47]. This should be contrasted with other properties; for example, the critical temperature of the Potts ferromagnet on an infinite-length lattice strip is zero for any width L_y regardless of how great.

For the honeycomb lattice, formal arguments give $q_c(hc) = (3 + \sqrt{5})/2 \simeq 2.618$ [48,43]. The honeycomb value is only formal because if q is not a positive integer, then one cannot use the expression of the partition function for the Potts antiferromagnet in terms of a sum of positive-definite Boltzmann factors and instead must use the formula [12] $Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{|E'|}$, where $v = e^K - 1$ with $K = \beta J$, where $\beta = (k_B T)^{-1}$ and J is the spin-

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spin coupling (which reduces to eq. (1.5) for the $T = 0$ antiferromagnet, where $K = -\infty$). However, for $-1 \leq v < 0$ and non-integral q, this does not, in general, define a Gibbs measure [48,49]. This non-integral property of $q_c(hc)$ for the two-dimensional honeycomb lattice and the associated subtleties make it especially useful to have explicit calculations of q_c values for finite-width strips of this lattice, to check that these are consistent with the expected behavior, according to which they approach the above value of $q_c(hc)$ as L_y gets large.

A second motivation related to this is that when one generalizes q from a positive integer to a complex variable, one sees that the point $q_c(\{G\})$ is the maximal point where a certain nonanalytic boundary $\mathcal{B}$ crosses the real axis (discussed further below); thus, another purpose of these exact calculations is to get further insight about this locus $\mathcal{B}$. As is well known, slight changes in coefficients of polynomials can have drastic changes on the positions of the zeros of the polynomials, which means that one must have the exact expression for $P(G, q)$ to study these zeros and their accumulation set $\mathcal{B}$. Since it was found that for (the $L_x \to \infty$ limit of) strips with free longitudinal boundary conditions, $\mathcal{B}$ does not necessarily cross the real axis, so that no $q_c(\{G\}, q)$ is defined [20,23], one cannot carry out this study in the same manner for these strips (although for sufficiently wide strips with free longitudinal boundary conditions, arcs on $\mathcal{B}$ have endpoints that are often close enough to the real axis to allow one to extrapolate and define an effective q_c).

A third motivation is that certain structural features of the transfer matrices can allow one to obtain general formulas that are applicable for arbitrarily large strip widths. We have already used this feature in Ref. [39], where we presented transfer matrices whose eigenvalues are $\lambda_{\Lambda, L_y, d}$ with degree $d = L_y - 1$ for several lattice strips with periodic boundary conditions, including those of the square and triangular lattices. Thus another purpose of exhibiting explicit transfer matrices for $0 \leq d \leq L_y - 2$ here is to make them available for a wider community, leading, hopefully, to some advances beyond those we have made with $d = L_y, L_y - 1$ in the construction of explicit formulas for $\lambda_{\Lambda, L_y, d}$ valid for arbitrary L_y. All of these degree-d sectors contribute to the partition function (here, the chromatic polynomial), so that they are of physical interest, although an elementary result is that the degeneracy per site W only depends on the (maximal eigenvalue in the) $d = 0$ sector.

The chromatic polynomial for the cyclic and Möbius strips of the square lattice were calculated (i) for $L_y = 2$ in [50] (see also [51]); (ii) for $L_y = 3$ cyclic in [29,30], and Möbius case in [31]; (iii) for $L_y = 4$ cyclic/Möbius in [35], (iv) for $L_y = 5$ cyclic/Möbius in [39]. The chromatic polynomials for the triangular lattice strips were calculated for (i) $L_y = 2$, cyclic/Möbius in [30], (ii) $L_y = 3, 4$ in [36]; and for the honeycomb lattice with (i) $L_y = 2$, cyclic/Möbius in [27], (ii) $L_y = 3$ in [10]. (Refs. [27,28] showed that the structure (1.4) holds more generally for cyclic strip graphs composed of iterated subgraphs that are not parts of

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regular lattices; we shall not need this degree of generality here.) We have proved that for these strips the set of eigenvalues are the same for cyclic (cyc.) and Möbius (Mb.) boundary conditions and have given the transformation rules for how the coefficients change for strips of the square lattice when one changes from cyclic to Möbius boundary conditions [9].

In our previous works on chromatic polynomials we have sometimes given transfer matrices (e.g. [39]) but often have expressed our results for the $\lambda_{\Lambda,L_y,d,j}$ in terms of generating functions or solutions to algebraic equations. In one respect this is a maximally compact way of presenting the results, since for a given set of $n_P(\Lambda, L_y, d)$ $\lambda_{\Lambda,L_y,d,j}$'s, it suffices to give the coefficients of the algebraic equation of degree $n_P(\Lambda, L_y, d)$ and for the generating function, it suffices to give $2n_P(\Lambda, L_y, d)$ coefficients, which are polynomials in q. Since the generating function is a rational function in q, it has the appeal that one always deals with polynomials. For the expression of the chromatic polynomial in terms of transfer matrices, since $T_{\Lambda,L_y,d}$ has dimension $n_P(\Lambda, L_y, d) \times n_P(\Lambda, L_y, d)$, one must specify a greater number of polynomials, namely $n_P(\Lambda, L_y, d)^2$ polynomials. As L_y increases, the number of expressions that one must give thus grows quadratically rather than linearly, as in the more compact approach using generating functions or algebraic equations for the $\lambda_{\Lambda,L_y,d}$. However, offsetting this disadvantage, the transfer matrix method has the advantage that some of the entries are rather simple, and, moreover, one can sometimes spot useful patterns in the matrices that expedite or confirm the construction of general formulas that are applicable for arbitrarily large strip widths L_y.

We mention some previous work. Matrix methods and related recursive linear algebraic techniques for calculating chromatic polynomials were used in early work [50–52] and more recent papers [53]- [55]. The coloring matrix technique of Ref. [52] was applied in Refs. [44]- [46] to obtain upper and lower bounds on W for various (infinite) two-dimensional lattices. This application relied upon the fact that these coloring matrices are non-negative, so that one could use the Perron-Frobenius theorem in analyzing the eigenvalues. Related recursive linear algebraic methods were used in Refs. [23–25] and in our subsequent papers on chromatic polynomials. Transfer matrices for chromatic polynomials were used in Refs. [41,42,56]. In Ref. [57] transfer matrices for the Potts model were developed and were applied there and in Refs. [58,59] to calculate chromatic polynomials for strips of the square and triangular lattices with free longitudinal boundary conditions and free or periodic transverse boundary conditions. These have been termed transfer matrices in the Fortuin-Kasteleyn representation (see eq. (1.5)). In Refs. [60,61], transfer matrix methods were used to calculate full Potts model partition functions with arbitrary q and v for strips of the square and triangular lattices with free longitudinal boundary conditions. Using transfer matrix methods together with the sieve methods of Refs. [53]- [55] (see also [62–64]), we have proved theorems

that determined $T_{\Lambda,L_y,d}$ for $d = L_y - 1$ and arbitrarily large L_y for several lattice strips with periodic boundary conditions [39].

Some remarks are in order here concerning an important respect in which the transfer matrices presented here differ from the conventional transfer matrices in statistical mechanics and coloring matrices in mathematical graph theory. Consider a strip graph, and let the spins on each transverse slice of this strip be labeled $\sigma_{x,y}$ where the longitudinal position x is fixed and the transverse position y varies. As conventionally defined, the transfer matrix of the (zero-field) Potts model at some temperature T is defined as the matrix $T_{x,x+1} = \langle\{\sigma_{x,y}\}|e^{-\beta\mathcal{H}}|\{\sigma_{x+1,y}\}\rangle$, where $\mathcal{H} = -J\sum_{\langle ij\rangle}\delta_{\sigma_i\sigma_j}$, $\sigma_i = 1, ..., q$ are the spin variables on each vertex and $\langle ij\rangle$ denotes pairs of adjacent vertices. Since there are q^{L_y} spin configurations on each slice, this is a $q^{L_y} \times q^{L_y}$ dimensional matrix. Although this dimension depends on q, the individual elements themselves do not explicitly depend on q. For physical temperatures $0 \le T \le \infty$ this matrix is non-negative, and at nonzero temperatures it is positive-definite. For $T > 0$, on a strip of finite width, for which this is a finite-dimensional matrix, one can then make powerful use of the Perron-Frobenius theorem on positive-definite matrices to conclude that there is a unique real positive eigenvalue of maximal magnitude. Indeed, this theorem provides a standard way to prove that the free energy of the Potts model or other spin model with short-range interactions on a strip of infinite length and finite width is analytic at all finite temperatures. Similarly, the coloring matrices used in Refs. [52,44–46] for studies of chromatic polynomials and the degeneracy per site W on various lattices are non-negative matrices, and this property was necessary for the upper and lower bounds derived in these papers. In contrast, the transfer matrices in the Fortuin-Kasteleyn representation are not positive-definite, and hence the Perron-Frobenius does not apply to them. The fact that it does not apply is crucial for the property that the locus $\mathcal{B}$ crosses the positive real axis and for the existence of $q_c(\{G\})$, since this locus occurs where there is a switch between at least two distinct eigenvalues of maximal magnitude and hence non-uniqueness of such eigenvalues.

From the exact expressions for $P(G,q)$, one can evaluate eq. (1.2) to get W and analyze the accumulation locus $\mathcal{B}$ of zeros of $P(G,q)$ in the complex q plane. We have done this for cyclic/Möbius strips in a number of previous works. These loci are different for strips with different boundary conditions, e.g., free, cylindrical, cyclic/Möbius, and toroidal. As we have shown, since the λ's are the same for the cyclic and Möbius strips of a given lattice, it follows that the locus $\mathcal{B}$ is the same for the infinite-length limits of cyclic and Möbius strips of a given lattice. In our previous work we have found a number of general properties of the locus $\mathcal{B}$, including the property that for strips of regular lattices with periodic or twisted periodic longitudinal boundary conditions the locus $\mathcal{B}$ is comprised of closed curves

that enclose various regions and pass through $q = 0$ and at a maximal real point q_c which depends on the lattice, as well as possible other intermediate points. The point $q_c(\{G\})$ is important since it separates the interval $q > q_c(\{G\})$ on the positive real q axis where the Potts model (with q extended from $\mathbb{Z}_+$ to $\mathbb{R}$) exhibits nonzero ground state entropy (which increases with q, asymptotically approaching $S_0 = k_B \ln q$ for large q) from the interval $0 \le q \le q_c(\{G\})$ in which S_0 has a different analytic form.

II. GENERAL STRUCTURE

Our results for chromatic polynomials are obtained as special cases, for the zero-temperature antiferromagnet, of general transfer matrices that we have calculated for the full temperature-dependent Potts model partition function on cyclic and Möbius lattice strips [65]. The latter results, and the general method, will be reported in a companion paper [66]. The motivations for presenting the special cases of these results for the zero-temperature Potts antiferromagnet (chromatic polynomial) have been given in the introduction. In addition to these physics motivations, a relevant point is that the transfer matrices for the chromatic polynomials are subtantially smaller in dimension and simpler in structure, depending only one one variable instead of two, than the transfer matrices for the full Potts model partition function. Another reason for separating these analyses of the full Potts model and the chromatic polynomial has to do with the determinants $det(T_{Z,\Lambda,L_y,d})$ and $det(T_{P,\Lambda,L_y,d})$; in both the case of the full Potts model and the chromatic polynomial, the individual eigenvalues $\lambda_{X,\Lambda.L_y,d}$ for most strip widths are not expressible in explicit algebraic form because the corresponding characteristic polynomials are of fifth order or higher. However, we have found a most interesting property, that the products of the eigenvalues that define the determinant $det(T_{Z,\Lambda,L_y,d})$ have a very simple form. When one specializes to $v = -1$, for all values of L_y and $d = 0, 1, ...L_y - 1$, except the lowest case, $L_y = 1$, some of these eigenvalues vanish, so that $det(T_{Z,\Lambda,L_y,d}) = 0$ at $v = -1$, reflecting the fact that the transfer matrices $T_{P,\Lambda,L_y,d}$ are of smaller dimension than $T_{Z,\Lambda,L_y,d}$. But in this $v = -1$ case, when one removes the zero columns and corresponding rows of $T_{Z,\Lambda,L_y,d}$ to form the transfer matrix $T_{P,\Lambda,L_y,d}$ for the chromatic polynomial, the resultant determinant is, in general, nonzero. And again we find a very interesting feature, namely that although the eigenvalues $\lambda_{P,\Lambda,L_y,d}$ themselves are roots of characteristic polynomials that are often of fifth or higher order, precluding solutions in terms of explicit algebraic expressions, the products that define $det(T_{P,\Lambda,L_y,d})$ are often quite simple, especially for the strips of the triangular and honeycomb lattices. This is another motivation for our presenting results for the chromatic polynomials separately from those for the full Potts model partition function.

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The chromatic polynomial for cyclic strips can be written in the form [7,9]

$$P(\Lambda, L_y \times m, cyc., q) = \sum_{d=0}^{L_y} c^{(d)} Tr((T_{\Lambda,L_y,d})^m) \tag{2.1}$$

where $T_{\Lambda,L_y,d}$ is the transfer matrix, with eigenvalues $\lambda_{\Lambda,L_y,d}$. We denote the characteristic polynomial of $T_{\Lambda,L_y,d}$ in the variable z as $CP(T_{\Lambda,L_y,d}, z)$. We have shown that the full transfer matrix T_{Λ,L_y} has a block structure formally specified by

$$T_{\Lambda,L_y} = \bigoplus_{d=0}^{L_y} \prod T_{\Lambda,L_y,d} \tag{2.2}$$

where the product $\prod T_{\Lambda,L_y,d}$ means a set of square blocks, each of dimension, $c^{(d)}$, of the form $\lambda_{\Lambda,L_y,d,j}$ times the identity matrix. The dimension of the total transfer matrix, i.e., the total number of eigenvalues $\lambda_{\Lambda,L_y,d,j}$ in eq. (1.4), counting multiplicities, is thus

$$dim(T_{\Lambda,L_y}) = \sum_{d=0}^{L_y} dim(T_{\Lambda,L_y,d}) = \sum_{d=0}^{L_y} c^{(d)} n_P(\Lambda, L_y, d) . \tag{2.3}$$

Each matrix $T_{\Lambda,L_y,d}$ is defined relative to a basis of d-color assignments to a path graph with L_y vertices corresponding to a transverse slice across the strip. Here one introduces partitions of the vertices in these path graphs. Using these as bases, we calculate the transfer matrix for each level (= degree) d, as discussed in our Ref. [66]. For the chromatic polynomials of the square and triangular lattices, we have the additional requirement that adjacent vertices cannot connect to each other. We list graphically all the possible partitions for $L_y = 2, 3, 4$ strips in Figs. 1 to 3, where white circles are the original L_y vertices and each black circle corresponds to a specific color assignment. We denote these partitions $\mathcal{P}_{L_y,d}$ for $2 \le L_y \le 5$ as follows:

$$\mathcal{P}_{2,0} = \{I\} , \qquad \mathcal{P}_{2,1} = \{\bar{2}; \bar{1}\} , \qquad \mathcal{P}_{2,2} = \{\bar{1}, \bar{2}\} \tag{2.4}$$

$$\mathcal{P}_{3,0} = \{I; 13\} , \qquad \mathcal{P}_{3,1} = \{\bar{3}; \bar{2}; \bar{1}; \overline{13}\}$$

$$\mathcal{P}_{3,2} = \{\bar{2}, \bar{3}; \bar{1}, \bar{3}; \bar{1}, \bar{2}\} , \qquad \mathcal{P}_{3,3} = \{\bar{1}, \bar{2}, \bar{3}\} \tag{2.5}$$

$$\mathcal{P}_{4,0} = \{I; 13; 14; 24\} , \qquad \mathcal{P}_{4,1} = \{\bar{4}; \bar{3}; \bar{2}; \bar{1}; 13, \bar{4}; \overline{13}; \overline{14}; \overline{24}; 24, \bar{1}\}$$

$$\mathcal{P}_{4,2} = \{\bar{3}, \bar{4}; \bar{2}, \bar{4}; \bar{1}, \bar{4}; \bar{2}, \bar{3}; \bar{1}, \bar{3}; \bar{1}, \bar{2}; \overline{13}, \bar{4}; \bar{1}, \overline{24}\}$$

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$$\mathcal{P}_{4,3} = \{\bar{2},\bar{3},\bar{4};\bar{1},\bar{3},\bar{4};\bar{1},\bar{2},\bar{4};\bar{1},\bar{2},\bar{3}\} \ , \qquad \mathcal{P}_{4,4} = \{\bar{1},\bar{2},\bar{3},\bar{4}\} \tag{2.6}$$

$$\mathcal{P}_{5,0} = \{I; 13; 14; 15; 24; 25; 35; 24, 15; 135\}$$

$$\mathcal{P}_{5,1} = \{\bar{5}; \bar{4}; \bar{3}; \bar{2}; \bar{1}; 13, \bar{5}; 13, \bar{4}; \overline{13}; 14, \bar{5}; \overline{14}; \overline{15}; 24, \bar{5}; \overline{24}; 24, \bar{1}; \overline{25}; 25, \bar{1}; \overline{35}; 35, \bar{2}; 35, \bar{1};$$

$$24, \overline{15}; \overline{135}\}$$

$$\mathcal{P}_{5,2} = \{\bar{4}, \bar{5}; \bar{3}, \bar{5}; \bar{2}, \bar{5}; \bar{1}, \bar{5}; \bar{3}, \bar{4}; \bar{2}, \bar{4}; \bar{1}, \bar{4}; \bar{2}, \bar{3}; \bar{1}, \bar{3}; \bar{1}, \bar{2}; 13, \bar{4}, \bar{5}; \overline{13}, \bar{5}; \overline{13}, \bar{4}; \overline{14}, \bar{5}; \overline{24}, \bar{5}; 24, \bar{1}, \bar{5};$$

$$\bar{1}, \overline{24}; \bar{1}, \overline{25}; \bar{2}, \overline{35}; \bar{1}, \overline{35}; 35, \bar{1}, \bar{2}\}$$

$$\mathcal{P}_{5,3} = \{\bar{3}, \bar{4}, \bar{5}; \bar{2}, \bar{4}, \bar{5}; \bar{1}, \bar{4}, \bar{5}; \bar{2}, \bar{3}, \bar{5}; \bar{1}, \bar{3}, \bar{5}; \bar{1}, \bar{2}, \bar{5}; \bar{2}, \bar{3}, \bar{4}; \bar{1}, \bar{3}, \bar{4}; \bar{1}, \bar{2}, \bar{4}; \bar{1}, \bar{2}, \bar{3}; \overline{13}, \bar{4}, \bar{5}; \bar{1}, \overline{24}, \bar{5};$$

$$\bar{1}, \bar{2}, \overline{35}\}$$

$$\mathcal{P}_{5,4} = \{\bar{2}, \bar{3}, \bar{4}, \bar{5}; \bar{1}, \bar{3}, \bar{4}, \bar{5}; \bar{1}, \bar{2}, \bar{4}, \bar{5}; \bar{1}, \bar{2}, \bar{3}, \bar{5}; \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$$

$$\mathcal{P}_{5,5} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} \tag{2.7}$$

where partitions are separated by a colon, and each overline corresponds to a color assignment. That is, for $d = 0$ there is no explicit color assignments; for $d = 1$, one vertex, or one connected set of vertices has its color specified; for $d = 2$, two vertices or two separate connected sets of vertices have their colors specified; and so forth for higher values of d. Therefore, in eqs. (2.1) to (2.4), there is no overline for $d = 0$, one overline for each partition of $d = 1$, two overlines for each partition of $d = 2$, etc.

The number of partitions $n_P(\Lambda, L_y, d)$ is the dimension of the transfer matrix $T_{\Lambda,L,d}$,

$$dim(T_{\Lambda,L_y,d}) = n_P(\Lambda, L_y, d) \ . \tag{2.8}$$

In Ref. [9] we determined the number $n_P(\Lambda, L_y, d)$ (labelled simply as $n_P(L_y, d)$) for the lattices $\Lambda = sq, tri$). Some special cases for $\Lambda = sq, tri$ are [9] as follows; analogous results for the honeycomb lattice were given in Ref. [10].

$$n_P(\Lambda, L_y, L_y) = 1 \tag{2.9}$$

$$n_P(\Lambda, L_y, L_y - 1) = L_y \tag{2.10}$$

$$n_P(\Lambda, L_y, 0) = n_P(\Lambda, L_y - 1, 1) = M_{L_y-1} \qquad (2.11)$$

where M_n is the Motzkin number in combinatorics, given by

$$M_n = \sum_{j=0}^{n} (-1)^j C_{n+1-j} \binom{n}{j} \qquad (2.12)$$

where $C_n = (n+1)^{-1} \binom{2n}{n}$ is the Catalan number. (We use the same symbol C_n for the circuit graph; the meaning will be clear from context). For reference, the first few Motzkin numbers are $M_n = 1, 2, 4, 9, 21, 51$ for $n = 1, ..., 6$, and we use the formal definition $M_0 = 1$.

Note that the number $n_P(tri, L_y, 0)$ is the same as the dimension of the corresponding transfer matrix for the triangular-lattice strip with free boundary conditions obtained in Ref. [57], while the transfer matrix for the free strip of the square lattice has a dimension (for which a general formula was given in Ref. [60]) which is smaller than $n_P(sq, L_y, 0)$ for $L_y \geq 4$. Consistent with this, for $L_y \geq 4$, we find that the characteristic polynomials for $T_{sq,4,0}$ and $T_{sq,5,0}$ factorize into parts such that one factor is the characteristic polynomial for the transfer matrix of the corresponding free strip, as will be seen below.
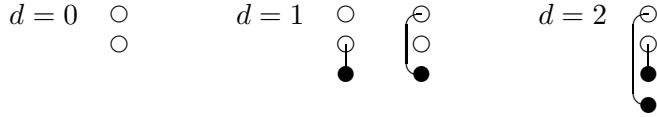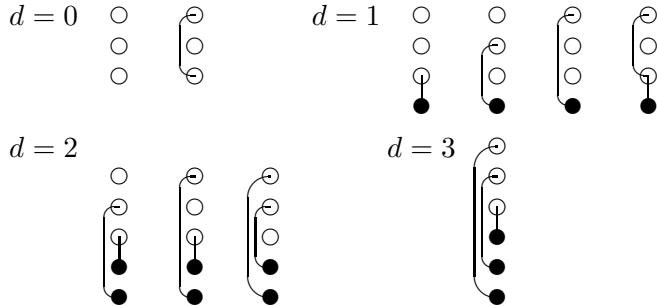

FIG. 1. Partitions for the $L_y = 2$ strip.

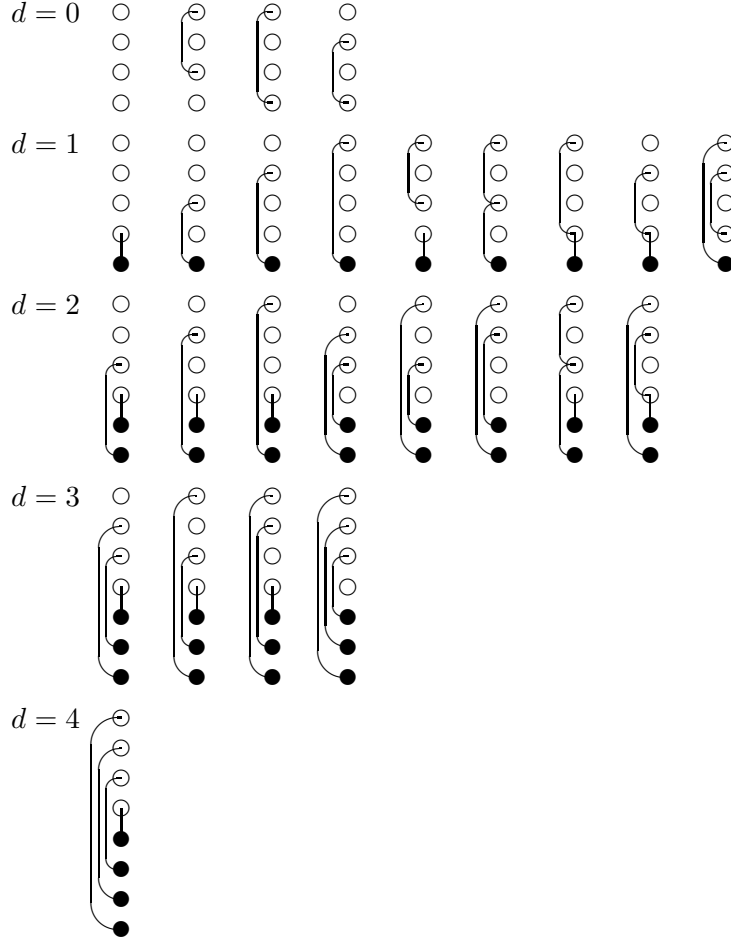FIG. 2. Partitions for the $L_y = 3$ strip.

FIG. 3. Partitions for the $L_y = 4$ strip.

As in earlier work, we define $N_{P,\Lambda,L_y,\lambda}$ as the total number of distinct eigenvalues of T_{Λ,L_y}, i.e. the sum of the dimensions of the submatrices $T_{\Lambda,L_y,d}$, modulo the multiplicity $c^{(d)}$:

$$N_{P,\Lambda,L_y,\lambda} = \sum_{d=0}^{L_y} n_P(\Lambda, L_y, d) \ . \tag{2.13}$$

Since we will use these results here, we include the relevant values in the Appendix. For cyclic or Möbius strips of the square and triangular lattices,

$$N_{P,\Lambda,L_y,\lambda} = 2(L_y - 1)! \sum_{j=0}^{[\frac{L_y}{2}]} \frac{(L_y - j)}{(j!)^2(L_y - 2j)!} \quad \text{for} \quad \Lambda = sq, tri \tag{2.14}$$

where $[\nu]$ denotes the integral part of ν.

For the corresponding strips of the honeycomb lattice, we calculated $n_P(hc, L_y, d)$ and $N_{P,hc,L_y,\lambda}$ in Ref. [10]. The results that are needed here are listed in the appendix. These

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correspond to the numbers of partitions which allow the connections between vertices 2 and 3, between vertices 4 and 5, etc. Therefore, in addition to the partitions given in eq. (2.5), the other partitions for the $L_y = 3$ strip of the honeycomb lattice are

$$\mathcal{P}_{hc,3,0} = \{23\} \ , \qquad \mathcal{P}_{hc,3,1} = \{\overline{23}; 23, \bar{1}\} \ , \qquad \mathcal{P}_{hc,3,2} = \{\bar{1}, \overline{23}\} \ . \tag{2.15}$$

In addition to the partitions given in eq. (2.6), the other partitions for the $L_y = 4$ strip of the honeycomb lattice are

$$\mathcal{P}_{hc,4,0} = \{23; 23, 14\} \ , \qquad \mathcal{P}_{hc,4,1} = \{23, \bar{4}; \overline{23}; 23, \bar{1}; 23, \overline{14}\}$$

$$\mathcal{P}_{hc,4,2} = \{\overline{23}, \bar{4}; 23, \bar{1}, \bar{4}; \bar{1}, \overline{23}\} \ , \qquad \mathcal{P}_{hc,4,3} = \{\bar{1}, \overline{23}, \bar{4}\} \tag{2.16}$$

In addition to the partitions given in eq. (2.7), the other partitions for the $L_y = 5$ strip of the honeycomb lattice are

$$\mathcal{P}_{hc,5,0} = \{23; 45; 13, 45; 23, 14; 23, 15; 23, 45; 145; 235; 245; 23, 145\}$$

$$\mathcal{P}_{hc,5,1} = \{23, \bar{5}; 23, \bar{4}; \overline{23}; 23, \bar{1}; \overline{45}; 45, \bar{3}; 45, \bar{2}; 45, \bar{1}; 13, \overline{45}; 45, \overline{13}; 14, 23, \bar{5}; 23, \overline{14}; 23, \overline{15};$$

$$23, \overline{45}; 45, \overline{23}; 23, 45, \bar{1}; \overline{145}; \overline{235}; 235, \bar{1}; \overline{245}; 245, \bar{1}; 23, \overline{145}\}$$

$$\mathcal{P}_{hc,5,2} = \{23, \bar{4}, \bar{5}; \overline{23}, \bar{5}; 23, \bar{1}, \bar{5}; \overline{23}, \bar{4}; 23, \bar{1}, \bar{4}; \bar{1}, \overline{23}; \bar{3}, \overline{45}; \bar{2}, \overline{45}; \bar{1}, \overline{45}; 45, \bar{2}, \bar{3}; 45, \bar{1}, \bar{3}; 45, \bar{1}, \bar{2};$$

$$\overline{13}, \overline{45}; 23, \overline{14}, \bar{5}; \overline{23}, \overline{45}; 23, \bar{1}, \overline{45}; 45, \bar{1}, \overline{23}; \bar{1}, \overline{235}; \bar{1}, \overline{245}\}$$

$$\mathcal{P}_{hc,5,3} = \{\overline{23}, \bar{4}, \bar{5}; 23, \bar{1}, \bar{4}, \bar{5}; \bar{1}, \overline{23}, \bar{5}; \bar{1}, \overline{23}, \bar{4}; \bar{2}, \bar{3}, \overline{45}; \bar{1}, \bar{3}, \overline{45}; \bar{1}, \bar{2}, \overline{45}; 45, \bar{1}, \bar{2}, \bar{3}; \bar{1}, \overline{23}, \overline{45}\}$$

$$\mathcal{P}_{hc,5,4} = \{\bar{1}, \overline{23}, \bar{4}, \bar{5}; \bar{1}, \bar{2}, \bar{3}, \overline{45}\} \ . \tag{2.17}$$

For cyclic strips of the square and triangular lattices, general properties include first [9]

$$dim(T_{\Lambda,L_y}) = P(\Lambda, L_y \times m, cyc., q)_{m=0} = \sum_{d=0}^{L_y} c^{(d)} n_P(\Lambda, L_y, d)$$

$$= P(\mathrm{Tree}_{L_y}, q) = q(q-1)^{L_y-1} \ , \quad \Lambda = sq, tri \tag{2.18}$$

where Tree_{L_y} denotes a tree graph (here a path graph) with L_y vertices; and

$$Tr(T_{\Lambda,L_y}) = P(\Lambda, L_y \times m, cyc., q)_{m=1} = \sum_{d=0}^{L_y} c^{(d)} Tr(T_{\Lambda,L_y,d}) = 0 \ , \quad \Lambda = sq, tri \ . \tag{2.19}$$

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For cyclic strips of the honeycomb lattice [10]

$$dim(T_{hc,L_y}) = P(hc, L_y \times m, cyc., q)_{m=0} = \sum_{d=0}^{L_y} c^{(d)} n_P(hc, L_y, d)$$

$$= \begin{cases} q\Big(q(q-1)\Big)^{\frac{L_y-1}{2}} & \text{for } L_y \text{ odd} \\ \Big(q(q-1)\Big)^{\frac{L_y}{2}} & \text{for } L_y \text{ even} \end{cases} \qquad (2.20)$$

Here we give another general result for these strips of the honeycomb lattice, governing the trace of the transfer matrix:

$$Tr(T_{hc,L_y}) = P(hc, L_y \times m, cyc., q)_{m=1} = \sum_{d=0}^{L_y} c^{(d)} Tr(T_{hc,L_y,d})$$

$$= P(\text{Tree}_{2L_y}, q) = q(q-1)^{2L_y-1} . \qquad (2.21)$$

This is proved by the same coloring matrix methods that we used in Ref. [9] and [10]. The reason for the difference relative to the corresponding formula (2.19) for the cyclic strips of the square and triangular lattices is that for $m = 1$ on those strips the longitudinal edges connect each vertex to itself, so the chromatic polynomial vanishes, but, in contrast, for the strip of the honeycomb lattice, the longitudinal edges have interior vertices of degree 2 associated with multiple edges. Because of this, and taking into account the elementary theorem that the chromatic polynomial of a graph remains unchanged if one replaces any edge by multiple copies of this edge, it follows that the resulting coloring is described by the chromatic polynomial $P(\text{Tree}_{2L_y}, q)$. For the reader's convenience in checking that our results satisfy these general trace formulas, we shall list traces of our transfer matrices below.

In general,

$$det(T_{\Lambda,L_y}) = \prod_{d=0}^{L_y} [det(T_{\Lambda,L_y,d})]^{c^{(d)}} . \qquad (2.22)$$

We next discuss Möbius strips. One of the major results of the present paper is a general formula, eq. (2.26) together with eqs. (2.23)-(2.25), for the partition function of the zero-temperature q-state Potts antiferromagnet on the Möbius strip of a regular lattice Λ of arbitrary width as well as arbitrary length. This formula involves a set of $\tilde{T}_{\Lambda,L_y,d}$ which occur once in each m-fold product (with the $T_{\Lambda,L_y,d}$ thus raised to the power $m - 1$) and which differ from the $T_{\Lambda,L_y,d}$ in the interchange of certain columns, encoding the Möbius property of reversed-orientation periodic longitudinal boundary conditions. In our previous papers

involving Möbius strips of the square and triangular lattices, we expressed the chromatic polynomial in a form analogous to (1.4) but with different sets of coefficients. In Ref. [9] we derived a set of transformations that governed how the coefficients for a certain d change when one changes the longitudinal boundary condition from cyclic to Möbius for the square-lattice strip. However, as noted there, this did not apply to the triangular-lattice strip; for that strip, if one expresses the chromatic polynomial in the same manner for the cyclic and Möbius cases, as a sum of m'th powers of eigenvalues $\lambda_{tri,L_y,d,j}$, then some of the coefficients become algebraic, rather than polynomial, functions of q. This was evident in the simplest nontrivial example, the $L_y = 2$ strip [30]. The general formalism that we introduce here for Möbius strips has the advantage that the coefficients stay in the set of $c^{(d)}$'s for triangular (and honeycomb) as well as square lattice strips. For $L_y \geq 3$, the $\tilde{T}_{\Lambda,L_y,d}$ matrices have eigenvalues that, in general, are not just related to those of $\tilde{T}_{\Lambda,L_y,d}$ by possible sign changes. In the limit of infinite length, $m \to \infty$, because only the $T_{\Lambda,L_y,d}$ matrices are raised to a power that goes to infinity, it is only the eigenvalues of these matrices that determine locus $\mathcal{B}$. Thus, the result of our previous papers, that the locus $\mathcal{B}$ is the same for the infinite-length limit of the strip of the lattice Λ for cyclic or Möbius boundary conditions, is again evident in our present formulation.

We proceed to the details. For Möbius strips one set of horizontal edges reverses the order of connection. This corresponds to the exchange of the pair of bases which switch to each other when the vertices reverse the order, i.e., the set of bases which do not have self-reflection symmetry with respect to the center of the tree with L_y vertices. For example, among the partitions for the $L_y = 2$ strip in Fig. 1, the first partition $\bar{2}$ and the second partition $\bar{1}$ in $\mathcal{P}_{2,1}$ must be exchanged under this reflection. The pairs of partitions for the $L_y = 3$ strip in Fig. 2 are the first partition $\bar{3}$ and the third partition $\bar{1}$ in $\mathcal{P}_{3,1}$, and the first partition $\bar{2}, \bar{3}$ and the third partition $\bar{1}, \bar{2}$ in $\mathcal{P}_{3,2}$. This corresponds to the exchange of these pairs of columns of $T_{\Lambda,L_y,d}$, and these matrices will be denoted as $\tilde{T}_{\Lambda,L_y,d}$ [66]. For general L_y we have the following changes of coefficients for the square, triangular and honeycomb lattices [9]:

$$c^{(0)} \to c^{(0)} \tag{2.23}$$

$$c^{(2k)} \to -c^{(k-1)} , \qquad 1 \leq k \leq \left[\frac{L_y}{2}\right] \tag{2.24}$$

$$c^{(2k+1)} \to c^{(k+1)} , \qquad 0 \leq k \leq \left[\frac{L_y - 1}{2}\right] . \tag{2.25}$$

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We thus obtain the important general formula, in terms of transfer matrices, for the chromatic polynomial for a Möbius strip of the lattice Λ, which is valid for arbitrary width L_y:

$$P(\Lambda, L_y \times m, Mb., q) = c^{(0)} Tr[(T_{\Lambda,L_y,0})^{m-1} \tilde{T}_{\Lambda,L_y,0}]$$

$$+ \sum_{d=0}^{[(L_y-1)/2]} c^{(d+1)} Tr[(T_{\Lambda,L_y,2d+1})^{m-1} \tilde{T}_{\Lambda,L_y,2d+1}]$$

$$- \sum_{d=1}^{[L_y/2]} c^{(d-1)} Tr[(T_{\Lambda,L_y,2d})^{m-1} \tilde{T}_{\Lambda,L_y,2d}] . \tag{2.26}$$

(where the factor $c^{(0)} = 1$ is displayed explicitly for uniformity).

For the square lattice, the eigenvalues of $\tilde{T}_{sq,L_y,d}$ are the same as those of $T_{sq,L_y,d}$ except for possible changes of signs. The number of eigenvalues with sign changes is the number of column-exchanges from $T_{sq,L_y,d}$ to $\tilde{T}_{sq,L_y,d}$. Denote the number of eigenvalues that are the same for $T_{sq,L_y,d}$ and $\tilde{T}_{sq,L_y,d}$ as $n_P(sq, L_y, d, +)$, and the number of eigenvalues with different signs as $n_P(sq, L_y, d, -)$. It is clear that

$$n_P(sq, L_y, d, +) + n_P(sq, L_y, d, -) = n_P(sq, L_y, d) . \tag{2.27}$$

Define

$$\Delta n_P(sq, L_y, d) \equiv n_P(sq, L_y, d, +) - n_P(sq, L_y, d, -) \tag{2.28}$$

which gives the number of partitions having self-reflection symmetry. For example, among the partitions for the $L_y = 2$ strip in Fig. 1, the partition I in $\mathcal{P}_{2,0}$ and the partition $\bar{1}, \bar{2}$ in $\mathcal{P}_{2,2}$ have self-reflection symmetry. Among the partitions for the $L_y = 3$ strip in Fig. 2, it includes the two partitions in $\mathcal{P}_{3,0}$, the second partition $\bar{2}$ and the fourth partition $\overline{13}$ in $\mathcal{P}_{3,1}$, the second partition $\bar{1}, \bar{3}$ in $\mathcal{P}_{3,2}$, and the partition in $\mathcal{P}_{3,3}$. We list $\Delta n_P(sq, L_y, d)$ for $1 \leq L_y \leq 10$ in Table III. The total number of these partitions for each L_y will be denoted as $\Delta N_{P,L_y}$, and we have $\Delta N_{P,2n-1} = \Delta N_{P,2n}$ for $n > 0$. The relations between $\Delta n_P(sq, L_y, d)$ are

$$\Delta n_P(sq, 2n - 1, 0) = \Delta n_P(sq, 2n, 0) , \qquad \text{for } 0 < n$$

$$\Delta n_P(sq, 2n - 1, 2m - 1) = \Delta n_P(sq, 2n, 2m) , \qquad \text{for } 1 \leq m \leq n$$

$$\Delta n_P(sq, 2n - 1, 2m) = \Delta n_P(sq, 2n, 2m - 1) , \qquad \text{for } 1 \leq m \leq n$$

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$$\Delta n_P(sq, 2n+1, 0) = 2\Delta n_P(sq, 2n, 0) + \Delta n_P(sq, 2n, 1) , \qquad \text{for } 0 < n$$

$$\Delta n_P(sq, 2n+1, m) = \Delta n_P(sq, 2n, m-1) + \Delta n_P(sq, 2n, m) + \Delta n_P(sq, 2n, m+1) ,$$

$$\text{for } 0 < m \le 2n+1$$

$$\Delta n_P(sq, 2n+1, 2m) = \Delta n_P(sq, 2n+1, 2m+1) , \qquad \text{for } 0 \le m \le n$$

$$\Delta n_P(sq, 2n, 0) = \Delta n_P(sq, 2n, 2)$$

$$\Delta n_P(sq, 2n, 2m-1) = \Delta n_P(sq, 2n, 2m+2) , \text{ for } 0 < m \le n-1 . \tag{2.29}$$

We also list $n_P(sq, L_y, d, +)$ and $n_P(sq, L_y, d, -)$ for $2 \le L_y \le 10$ in Table IV. Notice that $n_P(sq, L_y, 0, +)$ is the number of λ_{P,sq,FF,L_y} proved in [60]. Recall the numbers of $\lambda_{P,sq,L_y,j}$ for the Möbius strips of the square lattice with coefficients $\pm c^{(d)}$, defined as $n_{P,Mb}(L_y, d, \pm) \equiv n_{P,Mb}(sq, L_y, d, \pm)$, have been given in [9]. With the eqs. (2.23) to (2.25), the relations between $n_P(sq, L_y, d, \pm)$ and $n_{P,Mb}(sq, L_y, d, \pm)$ are

$$n_{P,Mb}(sq, L_y, 0, \pm) = n_P(sq, L_y, 0, \pm) + n_P(sq, L_y, 2, \mp)$$

$$n_{P,Mb}(sq, L_y, k, \pm) = n_P(sq, L_y, 2k-1, \pm) + n_P(sq, L_y, 2k+2, \mp) , \qquad 1 \le k \le \left[\frac{L_y+1}{2}\right] \tag{2.30}$$

We illustrate the application of this general formalism in the section where we give results for specific transfer matrices. For Möbius strips of the square lattice, the sum of the coefficients is given by

$$P(sq, L_y \times m, Mb., q)_{m=0} = c^{(0)}\Delta n_P(sq, L_y, 0) + \sum_{d=0}^{[(L_y-1)/2]} c^{(d+1)}\Delta n_P(sq, L_y, 2d+1)$$

$$- \sum_{d=1}^{[L_y/2]} c^{(d-1)}\Delta n_P(sq, L_y, 2d)$$

$$= c^{(0)}[\Delta n_P(sq, L_y, 0) - \Delta n_P(sq, L_y, 2)]$$

$$+ \sum_{d=1}^{[(L_y+1)/2]} c^{(d)}[\Delta n_P(sq, L_y, 2d-1) - \Delta n_P(sq, L_y, 2d+2)]$$

$$= \begin{cases} 0 & \text{for } L_y \text{ even} \\ P(\text{Tree}_{(L_y+1)/2}, q) & \text{for } L_y \text{ odd} \end{cases} \tag{2.31}$$

which agrees with Theorem 5 in [9] because of eqn. (2.30). This $m = 0$ equation also holds for the triangular lattice strips. If we take $m = 1$ in eqn. (2.26), the chromatic polynomial is zero for the square lattice with odd L_y or for the triangular lattice with any L_y. This is what we expect since these Möbius strips have at least one edge connect a vertex to itself.

For the honeycomb lattice with L_y even, the magnitudes of eigenvalues of $T_{hc,L_y,d}$ and $\tilde{T}_{hc,L_y,d}$ are again the same. We denote the number of eigenvalues that are the same as $n_P(hc, L_y, d, +)$, and the number of eigenvalues with different signs as $n_P(hc, L_y, d, -)$. Similarly to eqns. (2.27) and (2.28), we have

$$n_P(hc, L_y, d) = n_P(hc, L_y, d, +) + n_P(hc, L_y, d, -)$$

$$\Delta n_P(hc, L_y, d) \equiv n_P(hc, L_y, d, +) - n_P(hc, L_y, d, -) \ . \tag{2.32}$$

The values of $\Delta n_P(hc, L_y, d)$ for even L_y up to $L_y = 10$ are listed in Table V. The values $\Delta n_P(hc, L_y, d)$ can be obtained from $\Delta n_P(sq, L_y, d)$. For example,

$$\Delta n_P(hc, 2, d) = \Delta n_P(sq, 2, d)$$

$$\Delta n_P(hc, 4, d) = \Delta n_P(sq, 3, d) + \Delta n_P(sq, 4, d)$$

$$\Delta n_P(hc, 6, d) = \Delta n_P(sq, 4, d) + \Delta n_P(sq, 6, d)$$

$$\Delta n_P(hc, 8, d) = \Delta n_P(sq, 5, d) + \Delta n_P(sq, 6, d) + \Delta n_P(sq, 7, d) + \Delta n_P(sq, 8, d)$$

$$\Delta n_P(hc, 10, d) = \Delta n_P(sq, 6, d) + 2\Delta n_P(sq, 8, d) + \Delta n_P(sq, 10, d) \tag{2.33}$$

The relations between $\Delta n_P(hc, L_y, d)$ are

$$\Delta n_P(hc, 4n, 0) = 4\Delta n_P(hc, 4n - 2, 0) + 2\Delta n_P(hc, 4n - 2, 1) \ , \qquad \text{for } 0 < n$$

$$\Delta n_P(hc, 4n, 2m - 1) = n_P(hc, 4n, 2m)$$

$$= 2(\Delta n_P(hc, 4n - 2, 2m - 1) + \Delta n_P(hc, 4n - 2, 2m))$$

$$+\Delta n_P(hc, 4n - 2, 2m - 2) + \Delta n_P(hc, 4n - 2, 2m + 1) \ ,$$

$$\text{for } 1 \le m \le 2n$$

$$\Delta n_P(hc, 4n + 2, 0) = \Delta n_P(hc, 4n + 2, 2)$$

$$= \Delta n_P(hc, 4n, 0) + \Delta n_P(hc, 4n, 2) , \qquad \text{for } 1 \leq n$$

$$\Delta n_P(hc, 4n + 2, 2m - 1) = n_P(hc, 4n + 2, 2m + 2)$$

$$= \Delta n_P(hc, 4n, 2m - 1) + \Delta n_P(hc, 4n, 2m + 2) , \text{for } 1 \leq m \leq 2n$$

$$(2.34)$$

$n_P(hc, L_y, d, +)$ and $n_P(hc, L_y, d, -)$ for even $2 \leq L_y \leq 10$ are listed in Table VI. For Möbius strips of the honeycomb lattice with even L_y, the sum of the coefficients is given by

$$P(hc, L_y \times m, Mb., q)_{m=0} = c^{(0)} \Delta n_P(hc, L_y, 0) + \sum_{d=0}^{[(L_y-1)/2]} c^{(d+1)} \Delta n_P(hc, L_y, 2d + 1)$$

$$- \sum_{d=1}^{[L_y/2]} c^{(d-1)} \Delta n_P(hc, L_y, 2d)$$

$$= c^{(0)} [\Delta n_P(hc, L_y, 0) - \Delta n_P(hc, L_y, 2)]$$

$$+ \sum_{d=1}^{[(L_y+1)/2]} c^{(d)} [\Delta n_P(hc, L_y, 2d - 1) - \Delta n_P(hc, L_y, 2d + 2)]$$

$$= \begin{cases} 0 & \text{for } L_y/2 \text{ odd} \\ \left(q(q - 1)\right)^{L_y/4} & \text{for } L_y/2 \text{ even} \end{cases} \qquad (2.35)$$

It will be convenient to introduce the following polynomial:

$$D_n = \frac{P(C_n, q)}{q(q - 1)} = \sum_{j=0}^{n-2} (-1)^j \binom{n - 1}{j} q^{n-2-j} \qquad (2.36)$$

where $P(C_n, q) = (q - 1)^n + (q - 1)(-1)^n$ is the chromatic polynomial for the circuit graph with n vertices, C_n. Various properties of these polynomials were given in Refs. [25,26]. The first few D_n's are $D_1 = 0$, $D_2 = 1$, $D_3 = q - 2$, $D_4 = q^3 - 3q + 3$. In addition to the identities given in Refs. [25,26], another identity is

$$D_n + D_{n-1} = (q - 1)^{n-2} , n \geq 2 . \qquad (2.37)$$

Certain linear combinations of these polynomials also occur in the entries of the transfer matrices. These can be seen to arise as powers of basic linear factors such as $q - a$ plus integer constants; e.g., $q^2 - 4q + 5 = (q - 2)^2 + 1$, $q^2 - 6q + 10 = (q - 3)^2 + 1$, etc.

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We shall write out the entries of the transfer matrices explicitly if this can be done in the space available. In cases where this is not feasible, we shall express the entries in terms of the above polynomials, together with the shorthand notation

$$q_a = q - a \ , \quad q_{a,b} = (q-a)(q-b) \tag{2.38}$$

and additional shorthand defined in the last appendix, including the polynomials $F_{m,n}$, $G_{m,n}$, and others.

III. SOME TRANSFER MATRIX RESULTS FOR ARBITRARY L_y

A. $d = L_y$

For arbitrary L_y, eq. (2.9) shows that for $d = L_y$ the transfer matrix $T_{\Lambda,L_y,d=L}$ is one-dimensional, i.e., a scalar. We have previously proved that [39]

$$\lambda_{\Lambda,L_y,L_y} = (-1)^{L_y} \quad \text{for} \quad \Lambda = sq, tri \ . \tag{3.1}$$

Our results also led to the inference that [27,10]

$$\lambda_{\Lambda,L_y,L_y} = 1 \quad \text{for} \quad \Lambda = hc \ . \tag{3.2}$$

These results can also be seen to follow immediately from the transfer matrix method.

B. $d = L_y - 1$

1. Square lattice

For arbitrary L_y, the dimension of the transfer matrix is specified by eq. (2.10) as $n_P(\Lambda, L_y, L_y - 1) = L_y$ for $\Lambda = sq, tri$. For this case we have given the transfer matrices $T_{\Lambda,L_y,d=L_y-1}$ for cyclic strips of the square lattice in eqs. (7.1.1) and (7.1.4)-(7.1.6) of Ref. [39] and for cyclic strips of the triangular lattice in eqs. (7.3.1)-(7.3.4) of Ref. [39]. General formulas for eigenvalues and determinants were also given in Ref. [39]. Since we will need these here, we list them below:

$$\lambda_{sq,L_y,d=L_y-1,j} = (-1)^{L_y+1}(q - a_{sq,L_y,j}) \quad , \ 1 \le j \le L_y \tag{3.3}$$

where

$$a_{sq,L_y,j} = 1 + 4\cos^2\left(\frac{(L_y+1-j)\pi}{2L_y}\right) \quad , \ 1 \le j \le L_y \ . \tag{3.4}$$

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Hence, the determinant and trace of T_{sq,L_y,L_y-1} are

$$det(T_{sq,L_y,L_y-1}) = \prod_{j=1}^{L_y} (q - a_{sq,L_y,j}) \tag{3.5}$$

$$Tr(T_{sq,L_y,L_y-1}) = (-1)^{L_y+1}[2 + L_y(q-3)] . \tag{3.6}$$

Although our general result (3.3) exhibits an explicit factorization of the characteristic polynomial of T_{sq,L_y,L_y-1} into linear factors, this involves, in general, transcendental numbers (trigonometric functions) as coefficients. A few of these are of the form $\pm(q-k)$ where k is an integer; e.g., for $L_y = 2$, $d = 1$, one has $\lambda_{sq,2,1,j} = 1 - q, 3 - q$ and for $L_y = 3$, $d = 2$, one has $\lambda_{sq,3,2,j} = q - 1, q - 2, q - 4$. Some others involve terms of the form $\pm(q-a)$ where a is an algebraic number; e.g., for $L_y = 4$, $d = 3$, one has $\lambda_{4,3,j} = 1 - q, 3 - q, 3 \pm \sqrt{2} - q$. In Table VII we indicate the factorization properties of the various characteristic polynomials of $T_{sq,L_y,d}$ in factors with integer coefficients; for the illustrative case $L_y = 4, d = 3$, this is $CP(T_{sq,4,3}, z) = (z - 1 + q)(z - 3 + q)(z^2 - 2(3 - q)z + q^2 - 6q + 7)$, symbolized as $(1^2, 2)$ in the table. We observe that the factorizations often correspond to the numbers $n_P(sq, L_y, d, +)$ and $n_P(sq, L_y, d, -)$ in Table IV, although sometimes there are additional factorizations.

2. Triangular lattice

For T_{tri,L_y,L_y-1} we proved that [39]

$$det(T_{tri,L_y,L_y-1}) = (q - 2)^2(q - 3)^{L_y-2} \tag{3.7}$$

$$Tr(T_{tri,L_y,L_y-1}) = (-1)^{L_y+1}[3 + L_y(q-4)] . \tag{3.8}$$

Since a matrix and its transpose have the same eigenvalues, it is a convention which one is presented. The transfer matrices obtained directly from our calculation of the analogous matrices for the full Potts model are the transposes of those given in [39]. This just amounts to a different convention for the ordering of the basis of configurational states.

3. Honeycomb lattice

Here we give an exact determination of $T_{hc,L_y,d}$ for $d = L_y - 1$ which is valid for arbitrary L_y. This goes beyond our results in Ref. [39], which presented such formulas for the square and triangular lattices. We first observe that this matrix has dimension

$$dim(T_{hc,L_y,L_y-1}) = n_P(hc, L_y, L_y - 1) = \begin{cases} \frac{3L_y-1}{2} & \text{for } L_y \text{ odd} \\ \frac{3}{2}L_y - 1 & \text{for } L_y \text{ even} \end{cases} \tag{3.9}$$

This can be derived in a straightforward manner from results in [10]. It is convenient to work with two auxiliary sets of matrices, each of dimension $n_Z(hc, L_y, L_y - 1) = 2L_y - 1$, which are special cases of the transfer matrices for the full Potts model on strips of the honeycomb lattice, $S_{hc,L_y,L_y-1,j}$ for $j = 1, 2$. The elements of these matrices will be given below. We then form the product

$$S_{hc,L_y,L_y-1} = S_{hc,L_y,L_y-1,2}S_{hc,L_y,L_y-1,1} . \tag{3.10}$$

In this matrix certain columns are identically zero. The number of such zero columns is

$$n_Z(hc, L_y, L_y - 1) - n_P(hc, L_y, L_y - 1) = \begin{cases} \frac{L_y-1}{2} & \text{for } L_y \text{ odd} \\ \frac{L_y}{2} & \text{for } L_y \text{ even} \end{cases} \tag{3.11}$$

The transfer matrix T_{hc,L_y,L_y-1} is then the submatrix of S_{hc,L_y,L_y-1} formed by removing these columns with vanishing entries and the corresponding rows, which operation we denote with a subscript $red.$ for "reduced":

$$T_{hc,L_y,L_y-1} = S_{hc,L_y,L_y-1,red.} . \tag{3.12}$$

To determine T_{hc,L_y,L_y-1}, it thus suffices to give the $S_{hc,L_y,L_y-1,j}$ matrices for $j = 1, 2$. For the relevant range $L_y \geq 2$ where the strips of the honeycomb lattice are defined we calculate

$$(S_{hc,L_y,L_y-1,1})_{j,j} = 2 - q \quad \text{for} \quad 1 \leq j \leq L_y - 1 \tag{3.13}$$

$$(S_{hc,L_y,L_y-1,1})_{L_y,L_y} = \begin{cases} 2 - q & \text{for } L_y \text{ even} \\ 1 - q & \text{for } L_y \text{ odd} \end{cases} \tag{3.14}$$

$$(S_{hc,L_y,L_y-1,1})_{2j-1,2j} = (S_{hc,L_y,L_y-1,1})_{2j,2j-1} = 1 \quad \text{for} \quad 1 \leq j \leq [L_y/2] \tag{3.15}$$

$$(S_{hc,L_y,L_y-1,1})_{L_y+2j-1,L_y+2j-1} = 0 \quad \text{for} \quad 1 \leq j \leq [L_y/2] \tag{3.16}$$

$$(S_{hc,L_y,L_y-1,1})_{L_y+2j,L_y+2j} = 1 \quad \text{for} \quad 1 \leq j \leq [(L_y - 1)/2] \tag{3.17}$$

$$(S_{hc,L_y,L_y-1,1})_{2j-1,L_y+2j-1} = (S_{hc,L_y,L_y-1,1})_{2j,L_y+2j-1} = 0 \quad \text{for} \quad 1 \leq j \leq [L_y/2] \tag{3.18}$$

$$(S_{hc,L_y,L_y-1,1})_{2j,L_y+2j} = (S_{hc,L_y,L_y-1,1})_{2j+1,L_y+2j} = -1 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \qquad (3.19)$$

$$(S_{hc,L_y,L_y-1,1})_{L_y+2j-1,2j-1} = (S_{hc,L_y,L_y-1,1})_{L_y+2j-1,2j} = -1 \quad \text{for} \quad 1 \le j \le [L_y/2] \qquad (3.20)$$

$$(S_{hc,L_y,L_y-1,2})_{1,1} = 1 - q \qquad (3.21)$$

$$(S_{hc,L_y,L_y-1,2})_{j,j} = 2 - q \quad \text{for} \quad 2 \le j \le L_y - 1 \qquad (3.22)$$

$$(S_{hc,L_y,L_y-1,2})_{L_y,L_y} = \begin{cases} 2 - q & \text{for } L_y \text{ odd} \\ 1 - q & \text{for } L_y \text{ even} \end{cases} \qquad (3.23)$$

$$(S_{hc,L_y,L_y-1,2})_{2j,2j+1} = (S_{hc,L_y,L_y-1,1})_{2j+1,2j} = 1 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \qquad (3.24)$$

$$(S_{hc,L_y,L_y-1,2})_{L_y+2j-1,L_y+2j-1} = 1 \quad \text{for} \quad 1 \le j \le [L_y/2] \qquad (3.25)$$

$$(S_{hc,L_y,L_y-1,2})_{L_y+2j,L_y+2j} = 0 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \qquad (3.26)$$

$$(S_{hc,L_y,L_y-1,2})_{2j-1,L_y+2j-1} = (S_{hc,L_y,L_y-1,2})_{2j,L_y+2j-1} = -1 \quad \text{for} \quad 1 \le j \le [L_y/2] \qquad (3.27)$$

$$(S_{hc,L_y,L_y-1,2})_{2j,L_y+2j} = (S_{hc,L_y,L_y-1,2})_{2j+1,L_y+2j} = 0 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \qquad (3.28)$$

$$(S_{hc,L_y,L_y-1,2})_{L_y+2j,2j} = (S_{hc,L_y,L_y-1,1})_{L_y+2j,2j+1} = -1 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \qquad (3.29)$$

with all other elements equal to zero. Using our general formulas, we find that

$$det(T_{hc,L_y,L_y-1}) = (q-1)^{L_y+1} \quad \text{for odd} \quad L_y \ge 3 . \qquad (3.30)$$

For even L_y, $det(T_{hc,L_y,L_y-1})$ is proportional to $(q-1)^{L_y} F_{4,3} = (q-1)^{L_y}(q^2 - 4q + 5)$.

We next proceed to list our explicit calculations of transfer matrices. Since we have completely determined the transfer matrices in the cases $d = L_y$, where it is a scalar, and for the case $d = L_y - 1$, we do not list the former for each specific L_y.

IV. STRIPS OF THE SQUARE LATTICE

A. $L_y = 2$

We recall first that for $L_y = 1$, an elementary calculation yields $P(sq, 1 \times m, cyc., q) = (q-1)^m + (q-1)(-1)^m$ so $T_{sq,1,0} = q-1$. The chromatic polynomials $P(sq, 2 \times m, BC, q)$ for cyclic and Möbius BC's were given in Ref. [50] One has $n_P(\Lambda, 2, 0) = 1$, $n_P(\Lambda, 2, 1) = 2$, $n_P(\Lambda 2, 2) = 1$ for $\Lambda = sq$ (or tri), $T_{sq,2,0} = q^2 - 3q + 3$, and

$$T_{sq,2,1} = - \begin{pmatrix} q-2 & -1 \\ -1 & q-2 \end{pmatrix} \tag{4.1}$$

yielding the eigenvalues $\lambda_{sq,2,1,1} = 1 - q$ and $\lambda_{sq,2,1,1} = 3 - q$. The determinant $det(T_{sq,2,1}) = (q-1)(q-3)$ and trace $Tr(T_{sq,2,1}) = -2(q-2)$ follow as the $L_y = 2$ special case of our results in [39], viz., eqs. (3.5) and (3.6). For the present strip, the general methods discussed above yield, for the Möbius strip, the matrix $\tilde{T}_{sq,2,1}$ which has the first and second columns interchanged.

B. $L_y = 3$

The chromatic polynomials $P(sq, 3 \times m, BC, q)$ were given in [29,30] for cyclic BC's and in [31] for Möbius BC's. We find

$$T_{sq,3,0} = \begin{pmatrix} (q-2)(q^2 - 3q + 4) & q^2 - 4q + 5 \\ 1 & q-2 \end{pmatrix} \tag{4.2}$$

$$T_{sq,3,1} = \begin{pmatrix} -(q^2 - 4q + 5) & q-2 & -1 & 2-q \\ q-2 & -(q-2)^2 & q-2 & 1 \\ -1 & q-2 & -(q^2 - 4q + 5) & 2-q \\ 1 & 1 & 1 & q-2 \end{pmatrix} \tag{4.3}$$

$$T_{sq,3,2} = \begin{pmatrix} q-2 & -1 & 0 \\ -1 & q-3 & -1 \\ 0 & -1 & q-2 \end{pmatrix} \tag{4.4}$$

Although $T_{sq,3,0}$ is the same as the analogous transfer matrix for the free strip of width $L_y = 3$, $T_{sq,L_y,0}$ is larger than the free-strip transfer matrix for $L_y \geq 4$. Note that the upper

left-hand 3×3 block of $T_{sq,3,1}$ is symmetric. The matrix $\tilde{T}_{sq,3,1}$ is obtained by exchanging columns 1 and 3 of $T_{sq,3,1}$, and $\tilde{T}_{sq,3,2}$ is obtained by exchanging columns 1 and 3 of $T_{sq,3,2}$.

We have

$$det(T_{sq,3,0}) = (q-1)(q^3 - 6q^2 + 13q - 11) \tag{4.5}$$

$$det(T_{sq,3,1}) = -(q-1)(q-2)^2(q^4 - 9q^3 + 29q^2 - 40q + 22) \tag{4.6}$$

$$det(T_{sq,3,2}) = (q-1)(q-2)(q-4) \tag{4.7}$$

$$Tr(T_{sq,3,0}) = (q-2)(q^2 - 3q + 5) \tag{4.8}$$

$$Tr(T_{sq,3,1}) = -(3q^2 - 13q + 16) \ . \tag{4.9}$$

$Tr(T_{sq,3,2})$ is given by the $L_y = 3, d = 2$ special case of the general formula (7.1.28) of [39], given above as eq. (3.6). The characteristic polynomial $CP(T_{sq,3,1}, z)$ factorizes into polynomials of degree 1 and 3 in z. This is indicated in Table VII. The eigenvalue corresponding to the linear factor is $\lambda_{sq,3,1,1} = -(q-2)^2$ [29].

C. $L_y = 4$

The chromatic polynomials $P(sq, 4 \times m, BC, q)$ for BC=cyclic and Möbius were given in [35]. We find

$$T_{sq,4,0} = \begin{pmatrix} s_{21} & (q-2)p_6 & r_{20} & (q-2)p_6 \\ q-2 & (q-2)^2 & 3-q & 1 \\ -1 & 2-q & q^2 - 5q + 7 & 2-q \\ q-2 & 1 & 3-q & (q-2)^2 \end{pmatrix} \tag{4.10}$$

where the polynomials s_{21}, p_6, and r_{20} are defined in the appendix.

We have

$$det(T_{sq,4,0}) = (q-1)(q-3)(q^8 - 16q^7 + 112q^6 - 449q^5 + 1130q^4 - 1829q^3$$

$$+ 1858q^2 - 1084q + 279) \tag{4.11}$$

and

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$$Tr(T_{sq,4,0}) = q^4 - 7q^3 + 24q^2 - 45q + 36 \ . \tag{4.12}$$

As indicated in Table VII, the characteristic polynomial of $T_{sq,4,0}$ consists of factors of degree 1 and 3 in z. The eigenvalue which is the root of the linear factor is $\lambda_{sq,4,0,1} = (q-1)(q-3)$; the eigenvalues of the cubic factor are the same as the eigenvalues for the free strip of width $L_y = 4$ determined in Ref. [23]. The matrix $\tilde{T}_{sq,4,0}$ is obtained by exchanging columns 2 and 4 of $T_{sq,4,0}$.

$$T_{sq,4,1} = \begin{pmatrix}
-r_{13} & F_{4,3} & -q_2 & 1 & -G_{4,3} & q_2 & -G_{4,3} & -q_2^2 & q_2 \\
F_{4,3} & -q_2 F_{4,3} & q_2^2 & -q_2 & q_2 & -q_2^2 & q_3 & q_2 & -1 \\
-q_2 & q_2^2 & -q_2 F_{4,3} & F_{4,3} & -1 & q_2 & q_3 & -q_2^2 & q_2 \\
1 & -q_2 & F_{4,3} & -r_{13} & q_2 & -q_2^2 & -G_{4,3} & q_2 & -G_{4,3} \\
-1 & 0 & 0 & 0 & -q_2 & 0 & 0 & 0 & 0 \\
-1 & q_2 & q_2 & q_2 & -q_2 & q_2^2 & -q_3 & 1 & 1 \\
-1 & -1 & -1 & -1 & -q_2 & -q_2 & G_{4,3} & -q_2 & -q_2 \\
q_2 & q_2 & q_2 & -1 & 1 & 1 & -q_3 & q_2^2 & -q_2 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -q_2
\end{pmatrix} \tag{4.13}$$

The shorthand notation used in eq. (4.13) and below, such as $F_{m,n}$ and $G_{m,n}$, is defined in the last appendix. Note that the upper left-hand 4×4 block of $T_{sq,4,1}$ is symmetric. We calculate

$$det(T_{sq,4,1}) = (q-1)^3 (q-3)^5 (q^2 - 3q + 3)^2$$

$$\times \ (q^8 - 17q^7 + 125q^6 - 520q^5 + 1342q^4 - 2206q^3 + 2261q^2 - 1325q + 341) \tag{4.14}$$

$$Tr(T_{sq,4,1}) = -4q^3 + 27q^2 - 69q + 65 \ . \tag{4.15}$$

The matrix $\tilde{T}_{sq,4,1}$ is obtained by exchanging columns 1 and 4, 2 and 3, 5 and 9, and 6 and 8 of $T_{sq,4,1}$. It is ironic that although the chromatic polynomial is a special case of the full Potts model partition function, the determinants of the transfer matrices $T_{\Lambda,L_y,d}$ for the degree-d subspaces for these square-lattice strips, such as eq. (4.14), are more complicated than the simple expression that we found for the determinant of the transfer matrix $T_{Z,sq,L_y,d}$ in Ref. [66],

$$det(T_{Z,sq,L_y,d}) = v^{L_y[n_Z(L_y,d)-n_Z(L_y-1,d)]} \Big[(v+qr)^{L_y}(v+1)^{L_y-1} \Big]^{n_Z(L_y-1,d)} \tag{4.16}$$

where here $n_Z(L_y,d) \equiv n_Z(\Lambda, L_y, d)$ for $\Lambda = sq, tri$. Furthermore, the expression for $det(T_{sq,L_y,d})$ for these square-lattice strips gets more complicated as L_y increases, in contrast

to eq. (4.16) and also in contrast to the results that we find for $det(T_{\Lambda,L_y,d})$ for $\Lambda = tri, hc$ (see below).

As indicated in Table VII, the characteristic polynomial of $T_{sq,4,1}$ consists of factors of degree 4 and 5.

$$T_{sq,4,2} = \begin{pmatrix} F_{4,3} & 2-q & 1 & 0 & 0 & 0 & q-2 & 0 \\ 2-q & q^2-5q+6 & 3-q & 2-q & 1 & 0 & -1 & 0 \\ 1 & 3-q & p_8 & 1 & 3-q & 1 & q-2 & q-2 \\ 0 & 2-q & 1 & (q-2)^2 & 2-q & 0 & 0 & 0 \\ 0 & 1 & 3-q & 2-q & q_{2,3} & 2-q & 0 & -1 \\ 0 & 0 & 1 & 0 & 2-q & F_{4,3} & 0 & q-2 \\ -1 & -1 & -1 & 0 & 0 & 0 & 2-q & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 2-q \end{pmatrix} \tag{4.17}$$

where $F_{m,n}$ and p_8 are defined in the last appendix. The upper left-hand 6×6 block of $T_{sq,4,2}$ is symmetric. The matrix $\tilde{T}_{sq,4,2}$ is obtained by exchanging columns 1 and 6, 2 and 5, and 7 and 8 of $T_{sq,4,2}$. $CP(T_{sq,4,2}, z)$ has factors of degree 3 and 5. We have

$$det(T_{sq,4,2}) = (q-1)^2(q-3)^2(q^3 - 7q^2 + 15q - 11)$$

$$\times (q^7 - 16q^6 + 106q^5 - 378q^4 + 788q^3 - 967q^2 + 653q - 189) \tag{4.18}$$

$$Tr(T_{sq,4,2}) = 6q^2 - 29q + 38 \tag{4.19}$$

$$T_{sq,4,3} = -\begin{pmatrix} q-2 & -1 & 0 & 0 \\ -1 & q-3 & -1 & 0 \\ 0 & -1 & q-3 & -1 \\ 0 & 0 & -1 & q-2 \end{pmatrix} \tag{4.20}$$

From the general formulas (3.5) and (3.6), we have $det(T_{sq,4,3}) = (q-1)(q-3)(q^2-6q+7)$ and $Tr(T_{sq,4,3}) = -2(2q-5)$.

D. $L_y = 5$

The chromatic polynomials $P(sq, 5 \times m, BC, q)$ for BC=cyclic and Möbius were given in [39]. In addition to the general determination of T_{sq,L_y,L_y-1}, the transfer matrix for $T_{sq,5,3}$ was given in [39]. We display here our calculation of $T_{sq,5,0}$:

$$T_{sq,5,0} =$$

$$
\begin{pmatrix}
F_{4,3}r_{11} & s_{31} & s_{47} & s_{61} & q_2^2 p_7 & s_{47} & s_{31} & r_{19} & q_2 p_8 \\
F_{4,3} & q_2 F_{4,3} & -q_{2,3} & 2q-5 & q_2 & q_3 & q_2 & 1 & q_2^2 \\
-q_2 & -q_2^2 & q_2 G_{4,3} & -p_8 & -q_2^2 & q_3 & -1 & -q_2 & -q_2 \\
1 & q_2 & -G_{4,3} & r_{20} & q_2 & -G_{4,3} & q_2 & G_{4,3} & q_2^2 \\
q_2^2 & q_2 & -q_{2,3} & 2q_3 & q_2^3 & -q_{2,3} & q_2 & q_{2,3} & 1 \\
-q_2 & -1 & q_3 & -p_8 & -q_2^2 & q_2 G_{4,3} & -q_2^2 & -q_2 & -q_2 \\
F_{4,3} & q_2 & q_3 & 2q-5 & q_2 & -q_{2,3} & q_2 F_{4,3} & 1 & q_2^2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & q_2 & 0 \\
-1 & -q_2 & -q_3 & 5-2q & 1 & -q_3 & -q_2 & -1 & -q_2^2
\end{pmatrix}
$$

$$(4.21)$$

where the various p, r, s polynomials are defined in the appendix. The characteristic polynomial of $T_{sq,5,0}$ consists of factors of degree 2 and 7 in z. The eigenvalues that are the roots of the degree-7 factor are the same as those found in Ref. [23] for the corresponding $L_y = 5$ free strip of the square lattice.

The transfer matrices $T_{sq,5,d}$ for $d = 1, 2$ are both of dimension 21×21 and are too lengthy to present here; they are available from the authors. We find

$$det(T_{sq,5,0}) = -(q-1)^3(q-2)^2(q^6 - 13q^5 + 69q^4 - 191q^3 + 292q^2 - 236q + 79)$$

$$\times (q^{15} - 34q^{14} + 538q^{13} - 5259q^{12} + 35541q^{11} - 176036q^{10}$$

$$+ 660682q^9 - 1914798q^8 + 4324155q^7 - 7615130q^6 + 10381339q^5$$

$$- 10768339q^4 + 8235159q^3 - 4388527q^2 + 1459163q - 228580) . \tag{4.22}$$

The determinant $det(T_{sq,5,1})$ factorizes as $(q-1)^6(q-2)^2 P_{20}P_{29}$, where P_{20} and P_{29} are polynomials in q of degree 20 and 29 in q. $det(T_{sq,5,2})$ factorizes as $(q-1)^6(q-2)P_{19}P_{23}$, where P_{19} and P_{23} are polynomials in q of degree 19 and 23. For $d = 3$ we have

$$det(T_{sq,5,3}) = (q-1)^3(q-2)(q^8 - 20q^7 + 171q^6 - 817q^5 + 2389q^4$$

$$- 4387q^3 + 4954q^2 - 3155q + 869)(q^{11} - 29q^{10} + 375q^9 - 2853q^8 + 14188q^7$$

$$- 48439q^6 + 115934q^5 - 194762q^4 + 225461q^3 - 171683q^2 + 77617q - 15835) \tag{4.23}$$

and, from eq. (3.5),

$$det(T_{sq,5,4}) = (q-1)(q^2 - 5q + 5)(q^2 - 7q + 11) \tag{4.24}$$

We obtain the traces

$$Tr(T_{sq,5,0}) = q^5 - 9q^4 + 42q^3 - 120q^2 + 195q - 137 \tag{4.25}$$

$$Tr(T_{sq,5,1}) = -5q^4 + 46q^3 - 180q^2 + 346q - 269 \tag{4.26}$$

$$Tr(T_{sq,5,2}) = 10q^3 - 74q^2 + 198q - 189 \tag{4.27}$$

$$Tr(T_{sq,5,3}) = -10q^2 + 51q - 69 \tag{4.28}$$

with $Tr(T_{sq,5,4})$ being given by eq. (3.6).

V. STRIPS OF THE TRIANGULAR LATTICE

A. $L_y = 2$

The chromatic polynomials $P(tri, 2 \times m, BC, q)$ for BC=cyclic and Möbius were given in [30]. One has $\lambda_{tri,2,0} = (q-2)^2$, and, as a special case of our general result in eqs. (7.1.1) and (7.3.1)-(7.3.4) of [39],

$$T_{tri,2,1} = - \begin{pmatrix} q-3 & q-2 \\ -1 & q-2 \end{pmatrix} \tag{5.1}$$

with eigenvalues

$$\lambda_{tri,2,1,j} = \frac{1}{2}\left[5 - 2q \pm \sqrt{9 - 4q} \right] \tag{5.2}$$

Note that we use a different ordering convention for the basis configurations here than the one we used in Ref. [39], so that T_{tri,L_y,L_y-1} in terms of our present basis is the transpose of the corresponding matrix given in [39]. This has no effect on the eigenvalues, since the eigenvalues of a matrix A and its transpose A^T are the same. We have $det(T_{tri,2,1}) = (q-2)^2$ and $Tr(T_{tri,2,1}) = 5 - 2q$, in accordance with the general formulas (3.7) and (3.8). For the Möbius strip, the $\tilde{T}_{tri,2,1}$ matrix is obtained from $T_{tri,2,1}$ by interchanging the two columns, as was true for the square lattice strip. However, in contrast to the case of the square-lattice strip, where this just reverses the sign of one of the eigenvalues, here the $\tilde{T}_{tri,2,1}$ matrix has different eigenvalues than $T_{tri,2,1}$ (namely, $(1/2)(3-q \pm \sqrt{5q^2 - 22q + 25}\,)$). It is instructive

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to display how our general formula operates. For comparison, we first display the explicit chromatic polynomial for the cyclic strip [30]:

$$P(tri, 2 \times m, cyc., q) = c^{(0)}(\lambda_{tri,2,0})^m + c^{(1)}\left[(\lambda_{tri,2,1,1})^m + (\lambda_{tri,2,1,2})^m\right] + c^{(2)} \tag{5.3}$$

If one expresses the chromatic polynomial for the Möbius strip as a sum of powers of the same eigenvalues as for the cyclic strip, the result is [30]

$$P(tri, 2 \times m, Mb., q) = c^{(0)}(\lambda_{tri,2,0})^m - \frac{(q-1)(q-3)}{\sqrt{9-4q}}\left[(\lambda_{tri,2,1,1})^m - (\lambda_{tri,2,1,2})^m\right] - c^{(0)} \tag{5.4}$$

One sees that the coefficient of the second and third terms is no longer in the set of $c^{(d)}$'s and, indeed, is not a polynomial function of q. Of course, the square root in the denominator of this coefficient cancels so that, as must be true, the chromatic polynomial is a polynomial in q. As discussed in [31], this can be seen as a consequence of the theorem on symmetric functions of roots of a polynomial equation whose coefficients are polynomials in q; although these roots are not, in general polynomial functions of q, the symmetric functions can be expressed in terms of polynomials in q. In the present case, the second and third terms can be written in a manifestly symmetric manner by observing that the denominator of the coefficient is equal to $\lambda_{tri,2,1,1} - \lambda_{tri,2,1,2}$. With our general formula (2.26) we express the chromatic polynomial in a form that keeps the coefficients in the set of $c^{(d)}$ polynomials.

B. $L_y = 3$

The chromatic polynomials $P(tri, 3 \times m, BC, q)$ for BC=cyclic and Möbius were given in [36]. We have

$$T_{tri,3,0} = \begin{pmatrix} (q-2)(q^2-5q+7) & (q-3)^2 \\ 2-q & q-3 \end{pmatrix} \tag{5.5}$$

This yields the same eigenvalues that were earlier calculated for the free $L_y = 3$ strip in Ref. [23] and can be taken to be the transfer matrix for that strip.

$$T_{tri,3,1} = \begin{pmatrix} -(q^2-6q+10) & -(q-2)(q-3) & q-2 & 3-q \\ q-3 & -(q-2)(q-3) & -(q-2)(q-3) & 3-q \\ -1 & q-2 & -(q-2)(q-3) & 3-q \\ 2 & 2-q & 2-q & q-3 \end{pmatrix} \tag{5.6}$$

The characteristic polynomial $CP(T_{tri,3,0}, z)$ is a quadratic, while $CP(T_{tri,3,1}, z)$ has a linear and cubic factor, as indicated in Table VIII. We note that for odd L_y the factorizations

given in Table VIII correspond to the numbers $n_P(sq, L_y, d, +)$ and $n_P(sq, L_y, d, -)$ given in Table IV, but there are no nontrivial factorizations for even L_y.

As a special case of our general formulas given in eqs. (7.3.1)-(7.3.4) of [39], we have

$$T_{tri,3,2} = \begin{pmatrix} q-3 & q-4 & q-2 \\ -1 & q-4 & q-2 \\ 0 & -1 & q-2 \end{pmatrix} \tag{5.7}$$

We calculate

$$det(T_{tri,3,0}) = (q-2)^3(q-3) \tag{5.8}$$

$$det(T_{tri,3,1}) = -(q-2)^5(q-3)^2 \tag{5.9}$$

$$det(T_{tri,3,2}) = (q-2)^2(q-3) \tag{5.10}$$

$$Tr(T_{tri,3,0}) = q^3 - 7q^2 + 18q - 17 \tag{5.11}$$

$$Tr(T_{tri,3,1}) = -3q^2 + 17q - 25 \tag{5.12}$$

$$Tr(T_{tri,3,2}) = 3(q-3) \ . \tag{5.13}$$

C. $L_y = 4$

The chromatic polynomials $P(tri, 4 \times m, cyc., q)$ were given in [36]. The transfer matrices that we give below, together with our general procedure for obtaining $\tilde{T}_{\Lambda,L,d}$ from $T_{\Lambda,L,d}$, yields a new solution for $P(tri, 4 \times m, Mb., q)$. We find

$$T_{tri,4,0} = \begin{pmatrix} (q-2)(q-3)p_8 & (q-2)(q-3)(q-4) & (q-3)p_{15} & (q-3)p_{10} \\ -(q-2)(q-3) & (q-2)(q-3) & 2(3-q) & 3-q \\ q-2 & (q-2)(q-3) & q^2-7q+13 & 3-q \\ -(q-2)(q-3) & (q-2)(q-3) & (q-3)(q-5) & (q-3)^2 \end{pmatrix} \tag{5.14}$$

where p_8, etc. are defined in the appendix. This yields the same characteristic polynomial and eigenvalues as those obtained for the free $L_y = 4$ strip in Ref. [23] and can be taken as the transfer matrix for that strip. We have

$$det(T_{tri,4,0}) = (q-2)^6(q-3)^4 \tag{5.15}$$

$$Tr(T_{tri,4,0}) = q^4 - 10q^3 + 42q^2 - 88q + 76 . \tag{5.16}$$

(We recall that the actual form of the matrix is basis-dependent; for example, for the free $L_y = 4$ strip, a different matrix which, however, also has the same characteristic polynomial and eigenvalues originally calculated in Ref. [23], was given in Ref. [59].)

$$T_{tri,4,1} = \begin{pmatrix} -r_{34} & -q_2p_{10} & q_{2,3} & -q_2 & -q_{3,5} & -q_{2,3} & -p_{13} & -q_3^2 & q_3 \\ p_{10} & -q_2p_{10} & -q_2q_3^2 & q_{2,3} & q_3 & -q_{2,3} & 2q_3 & -q_3^2 & q_3 \\ -q_3 & q_{2,3} & -q_2q_3^2 & -q_2p_{10} & q_3 & -q_{2,3} & -p_{13} & -q_3^2 & -q_{3,4} \\ 1 & -q_2 & q_{2,3} & -q_2p_{10} & q_3 & -q_{2,3} & -p_{13} & q_3 & -q_{3,4} \\ q_4 & q_2 & 0 & 0 & -q_3 & 0 & 0 & 0 & 0 \\ -2 & 2q_2 & -q_{2,3} & -q_{2,3} & -q_3 & q_{2,3} & -2q_3 & -q_3 & -q_3 \\ -2 & q_2 & q_2 & q_2 & -2q_3 & q_{2,3} & p_{13} & -q_3 & -q_3 \\ 2q_3 & -q_{2,3} & -q_{2,3} & 2q_2 & -2q_3 & q_{2,3} & p_{14} & q_3^2 & -2q_3 \\ 0 & 0 & 0 & q_2 & 0 & 0 & 1 & 0 & -q_3 \end{pmatrix} \tag{5.17}$$

We calculate

$$det(T_{tri,4,1}) = (q-2)^{12}(q-3)^8 \tag{5.18}$$

$$Tr(T_{tri,4,1}) = -2(q-3)(2q^2 - 12q + 21) \tag{5.19}$$

$$T_{tri,4,2} = \begin{pmatrix} p_{10} & q_{3,4} & -q_4 & q_{2,3} & -q_2 & 0 & q_3 & 0 \\ -q_3 & q_{3,4} & p_{14} & q_{2,3} & q_{2,4} & -q_2 & q_3 & q_3 \\ 1 & -q_4 & p_{14} & -q_2 & q_{2,4} & -q_2 & q_3 & q_3 \\ 0 & -q_3 & -q_4 & q_{2,3} & q_{2,4} & q_{2,3} & 0 & q_3 \\ 0 & 1 & -q_4 & -q_2 & q_{2,4} & q_{2,3} & 0 & q_3 \\ 0 & 0 & 1 & 0 & -q_2 & q_{2,3} & 0 & q_3 \\ -2 & q_4 & q_4 & q_2 & q_2 & 0 & -q_3 & 0 \\ 0 & 0 & -2 & 0 & q_2 & q_2 & 0 & -q_3 \end{pmatrix} \tag{5.20}$$

$$det(T_{tri,4,2}) = (q-2)^8(q-3)^6 \tag{5.21}$$

$$Tr(T_{tri,4,2}) = 6q^2 - 38q + 62 \tag{5.22}$$

As a special case of our general result in [39], we have

$$T_{tri,4,3} = - \begin{pmatrix} q-3 & q-4 & q-4 & q-2 \\ -1 & q-4 & q-4 & q-2 \\ 0 & -1 & q-4 & q-2 \\ 0 & 0 & -1 & q-2 \end{pmatrix} \tag{5.23}$$

The determinant and trace are the $L_y = 4$ special case of our general formulas (3.7) and (3.8), viz., $det(T_{tri,4,3}) = (q-2)^2(q-3)^2$ and $Tr(T_{tri,4,3}) = 13 - 4q$.

D. $L_y = 5$

We present here new results for $P(tri, 5 \times m, BC, q)$ for BC=cyclic and Möbius. For $d = 0$ we calculate

$T_{tri,5,0} =$

$$
\begin{pmatrix}
q_2 s_{82} & q_{2,3} p_{13} & q_2 r_{58} & s_{173} & q_{2,3} p_{13} & s_{148} & q_3 r_{34} & q_3 p_{17} & q_5 q_3^2 \\
-q_2 p_{10} & q_2 q_3^2 & -2q_{2,3} & -q_4^2 & -q_{2,3} & -p_{13} & -q_{3,4} & -q_3 & q_3^2 \\
q_{2,3} & q_2 q_3^2 & q_2 p_{13} & -q_4^2 & -q_{2,3} & 2q_3 & q_3 & -q_3 & q_3^2 \\
-q_2 & -q_{2,3} & q_2 p_{13} & r_{47} & -q_{2,3} & -p_{13} & q_3 & q_{3,5} & q_3^2 \\
-q_2 q_3^2 & q_2 q_3^2 & q_2 q_4^2 & -q_4(2q-7) & q_2 q_3^2 & -2q_3^2 & -q_3^2 & q_{3,4} & q_3^2 \\
q_{2,3} & -q_{2,3} & q_2 p_{13} & r_{49} & q_2 q_3^2 & q_3 p_{13} & -q_3^2 & q_3(2q-7) & q_3^2 \\
-q_2 p_{10} & -q_{2,3} & -2q_{2,3} & -q_4(2q-7) & q_2 q_3^2 & r_{46} & q_3 p_{10} & q_{3,4} & q_3^2 \\
0 & 0 & -q_2 & -q_4 & 0 & 0 & 0 & q_3 & 0 \\
2q_2 & q_{2,3} & 2q_{2,3} & q_{4,5} & q_{2,3} & p_{14} & -2q_3 & 2q_3 & -q_3^2
\end{pmatrix}
$$

$$(5.24)$$

where the various p, r, and s polynomials are defined in the appendix. We find

$$det(T_{tri,5,0}) = -(q-2)^{14}(q-3)^{12} \tag{5.25}$$

$$Tr(T_{tri,5,0}) = (q-3)(q^4 - 10q^3 + 46q^2 - 112q + 118) . \tag{5.26}$$

The characteristic polynomial of $T_{tri,5,0}$ consists of factors of degree 2 and 7.

The transfer matrices $T_{tri,5,d}$ for $d = 1, 2$ are both of dimension 21×21 and are too lengthy to present here; they are available from the authors. However, we do note the simple results

$$det(T_{tri,5,1}) = (q-2)^{30}(q-3)^{27} \tag{5.27}$$

$$Tr(T_{tri,5,1}) = -5q^4 + 62q^3 - 306q^2 + 708q - 642 \tag{5.28}$$

$$det(T_{tri,5,2}) = (q-2)^{25}(q-3)^{24} \tag{5.29}$$

$$Tr(T_{tri,5,2}) = 10q^3 - 98q^2 + 332q - 387 . \tag{5.30}$$

Factorizations of $CP(T_{tri,5,j}, z)$, $j = 1, 2$, are given in Table VIII.

For $d = 3$ we calculate

$T_{tri,5,3} =$

$$\begin{pmatrix}
-p_{10} & -q_{3,4} & q_4 & -q_{3,4} & q_4 & 0 & -q_{2,3} & q_2 & 0 & 0 & -q_3 & 0 & 0 \\
q_3 & -q_{3,4} & -p_{14} & -q_{3,4} & -q_4^2 & q_4 & -q_{2,3} & -q_{2,4} & q_2 & 0 & -q_3 & -q_3 & 0 \\
-1 & q_4 & -p_{14} & q_4 & -q_4^2 & q_4 & q_2 & -q_{2,4} & q_2 & 0 & -q_3 & -q_3 & 0 \\
0 & q_3 & q_4 & -q_{3,4} & -q_4^2 & -p_{14} & -q_{2,3} & -q_{2,4} & -q_{2,4} & q_2 & 0 & -q_3 & -q_3 \\
0 & -1 & q_4 & q_4 & -q_4^2 & -p_{14} & q_2 & -q_{2,4} & -q_{2,4} & q_2 & 0 & -q_3 & -q_3 \\
0 & 0 & -1 & 0 & q_4 & -p_{14} & 0 & q_2 & -q_{2,4} & q_2 & 0 & -q_3 & -q_3 \\
0 & 0 & 0 & q_3 & q_4 & q_4 & -q_{2,3} & -q_{2,4} & -q_{2,4} & -q_{2,3} & 0 & 0 & -q_3 \\
0 & 0 & 0 & -1 & q_4 & q_4 & q_2 & -q_{2,4} & -q_{2,4} & -q_{2,3} & 0 & 0 & -q_3 \\
0 & 0 & 0 & 0 & -1 & q_4 & 0 & q_2 & -q_{2,4} & -q_{2,3} & 0 & 0 & -q_3 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & q_2 & -q_{2,3} & 0 & 0 & -q_3 \\
2 & -q_4 & -q_4 & -q_4 & -q_4 & 0 & -q_2 & -q_2 & 0 & 0 & q_3 & 0 & 0 \\
0 & 0 & 2 & 0 & -q_4 & -q_4 & 0 & -q_2 & -q_2 & 0 & 0 & q_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -q_2 & -q_2 & 0 & 0 & q_3
\end{pmatrix}$$

$$(5.31)$$

We calculate

$$det(T_{tri,5,3}) = (q-2)^{11}(q-3)^{12} \tag{5.32}$$

$$Tr(T_{tri,5,3}) = -10q^2 + 67q - 115 . \tag{5.33}$$

The characteristic polynomial of $T_{tri,5,3}$ has factors of degree 5 and 8.

For $d = 4$, our general theorem in [39] yields

$$T_{tri,5,4} = \begin{pmatrix}
q-3 & q-4 & q-4 & q-4 & q-2 \\
-1 & q-4 & q-4 & q-4 & q-2 \\
0 & -1 & q-4 & q-4 & q-2 \\
0 & 0 & -1 & q-4 & q-2 \\
0 & 0 & 0 & -1 & q-2
\end{pmatrix} \tag{5.34}$$

In accordance with our general results (3.7) and (3.8), we have

$$det(T_{tri,5,4}) = (q-2)^2(q-3)^3 \tag{5.35}$$

$$Tr(T_{tri,5,4}) = 5q - 17 . \tag{5.36}$$

The characteristic polynomial of $T_{tri,5,4}$ has factors of degree 2 and 3.

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VI. STRIPS OF THE HONEYCOMB LATTICE

A. $L_y = 2$

The chromatic polynomials $P(hc, 2 \times m, BC, q)$ for BC=cyclic and Möbius were given in [27]. For $d = 0$, $T_{hc,2,0}$ is a scalar, $T_{hc,2,0} = \lambda_{hc,2,0} = D_6 = q^4 - 5q^3 + 10q^2 - 10q + 5$. For $d = 1$, from our general formulas above, we obtain

$$S_{hc,2,1,1} = \begin{pmatrix} 2-q & 1 & 0 \\ 1 & 2-q & 0 \\ -1 & -1 & 0 \end{pmatrix} \tag{6.1}$$

$$S_{hc,2,1,2} = \begin{pmatrix} 1-q & 0 & -1 \\ 0 & 1-q & -1 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.2}$$

whence

$$T_{hc,2,1} = \begin{pmatrix} D_4 & -D_3 \\ -D_3 & D_4 \end{pmatrix} = \begin{pmatrix} q^2 - 3q + 3 & 2-q \\ 2-q & q^2 - 3q + 3 \end{pmatrix} \tag{6.3}$$

yielding the relevant special case of the eigenvalues given in eq. (6), (7) of [27], $\lambda_{hc,2,1,1} = D_4 - D_3 = F_{4,3} = q^2 - 4q + 5$ and $\lambda_{hc,2,1,2} = D_4 + D_3 = (q-1)^2$. We note that

$$det(T_{hc,2,1}) = (q-1)^2(q^2 - 4q + 5) \tag{6.4}$$

and $Tr(T_{hc,2,1}) = 2D_4$

B. $L_y = 3$

Further,

$$T_{hc,3,0} = \begin{pmatrix} q_{1,2}s_7 & q_2 s_{3,5} & t_{5,5} \\ q_1 & F_{4,3} & -q_3 \\ -F_{6,2} & -q_2 p_4 & -F_{5,4} \end{pmatrix} \tag{6.5}$$

so that

$$det(T_{hc,3,0}) = (q-1)^4(q-2)^2 \tag{6.6}$$

$$Tr(T_{hc,3,0}) = q^6 - 8q^5 + 28q^4 - 56q^3 + 71q^2 - 58q + 26 \ . \tag{6.7}$$

For $d = 1$,

$$T_{hc,3,1} = \begin{pmatrix} q_{1,2}p_4 & -q_1F_{4,3} & 1 & r_{13} & -G_{4,3} & q_2p_6 \\ -q_1q_2^2 & q_{1,2}D_4 & -D_5 & -q_{2,3} & q_3D_4 & -2q_2^2 \\ q_{1,2} & -q_1D_4 & F_{6,5} & -q_{2,3} & q_3D_4 & q_2F_{4,3} \\ -q_{1,2} & q_1 & 0 & -q_3 & 1 & -q_2 \\ -q_{1,2} & q_1D_4 & D_5 & -2q_2 & 2D_4 & q_{2,3} \\ q_1 & q_1 & -1 & -q_3 & -q_3 & F_{4,3} \end{pmatrix} \tag{6.8}$$

from which it follows that

$$det(T_{hc,3,1}) = (q-1)^8(q-2)^2 \tag{6.9}$$

$$Tr(T_{hc,3,1}) = 3q^4 - 18q^3 + 46q^2 - 60q + 37 . \tag{6.10}$$

For $d = 2$, applying our general formulas above, we calculate

$$S_{hc,3,2,1} = \begin{pmatrix} 2-q & 1 & 0 & 0 & 0 \\ 1 & 2-q & 0 & 0 & -1 \\ 0 & 0 & 1-q & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.11}$$

$$S_{hc,3,2,2} = \begin{pmatrix} 1-q & 0 & 0 & -1 & 0 \\ 0 & 2-q & 1 & -1 & 0 \\ 0 & 1 & 2-q & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{pmatrix} \tag{6.12}$$

The factorizations for $\Lambda = hc$ are given in Table IX. We note that for even L_y, the factor-
izations in this table correspond to the numbers $n_P(hc, L_y, d, +)$ and $n_P(hc, L_y, d, -)$ given
in Table II, when these numbers can be defined, but there are no nontrivial factorizations
for odd L_y. We calculate

$$det(T_{hc,3,2}) = (q-1)^4 \tag{6.13}$$

$$Tr(T_{hc,3,2}) = 3q^2 - 10q + 12 \tag{6.14}$$

C. $L_y = 4$

For the 6×6 dimensional matrix $T_{hc,4,0}$ we calculate

$$det(T_{hc,4,0}) = 2(q-1)^8(q-2)^2(q^2 - 5q + 7)(2q^6 - 20q^5 + 83q^4 - 185q^3 + 239q^2 - 175q + 60)$$

(6.15)

$$Tr(T_{hc,4,0}) = q^8 - 11q^7 + 55q^6 - 165q^5 + 333q^4 - 480q^3 + 503q^2 - 362q + 142 .$$
(6.16)

The characteristic polynomial has factors of degree 1 and 5, as indicated in Table IX. The eigenvalues are

$$\lambda_{hc,4,0,1} = (q-1)^2(q^2 - 5q + 7)$$

(6.17)

and the roots of the factor of degree 5.

For the 13×13 dimensional matrix $T_{hc,4,1}$ we calculate

$$det(T_{hc,4,1}) = 4(q-1)^{16}(q-2)^2(q^8 - 15q^7 + 100q^6 - 389q^5 + 974q^4 - 1623q^3$$

$$+1774q^2 - 1171q + 360)(q^{12} - 20q^{11} + 185q^{10} - 1048q^9 + 4061q^8$$

$$-11385q^7 + 23784q^6 - 37468q^5 + 44352q^4 - 38630q^3 + 23631q^2 - 9200q + 1750)$$

(6.18)

$$Tr(T_{hc,4,1}) = 4q^6 - 36q^5 + 147q^4 - 351q^3 + 529q^2 - 489q + 229 .$$

(6.19)

The characteristic polynomial factorizes into factors of degree 5 and 8.

For the 11×11 dimensional matrix $T_{hc,4,2}$ we find

$$det(T_{hc,4,2}) = (q-1)^{12}(q^6 - 10q^5 + 44q^4 - 108q^3 + 160q^2 - 141q + 60)$$

$$\times(2q^8 - 29q^7 + 186q^6 - 690q^5 + 1634q^4 - 2564q^3 + 2647q^2 - 1670q + 500)$$

(6.20)

$$Tr(T_{hc,4,2}) = 6q^4 - 39q^3 + 110q^2 - 157q + 104$$

(6.21)

For $T_{hc,4,3}$, applying our general formula (3.12), we calculate

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$$T_{hc,4,3} = \begin{pmatrix} D_4 & 2-q & 0 & 0 & 0 \\ 3-q & F_{4,3} & 2-q & 1 & q-3 \\ 1 & 2-q & F_{4,3} & 3-q & q-3 \\ 0 & 0 & 2-q & D_4 & 0 \\ -1 & q-2 & q-2 & -1 & 2 \end{pmatrix} \tag{6.22}$$

whence

$$det(T_{hc,4,3}) = 2(q-1)^4(q^2 - 4q + 5) \tag{6.23}$$

$$Tr(T_{hc,4,3}) = 2(2q^2 - 7q + 9) \tag{6.24}$$

The eigenvalues are

$$\lambda_{hc,4,3,1} = (q-1)^2 \tag{6.25}$$

$$\lambda_{hc,4,3,2} = F_{4,3} = q^2 - 4q + 5 \tag{6.26}$$

and the three roots of

$$z^3 - 2(q^2 - 4q + 6)z^2 + (q^4 - 8q^3 + 26q^2 - 36q + 21)z - 2(q-1)^2 = 0 \ . \tag{6.27}$$

D. $L_y = 5$

We have calculated the $T_{hc,5,d}$ and find

$$det(T_{hc,5,0}) = (q-1)^{36}(q-2)^{12} \tag{6.28}$$

$$Tr(T_{hc,5,0}) = q^{10} - 14q^9 + 91q^8 - 364q^7 + 1007q^6 - 2058q^5 + 3232q^4$$

$$-3950q^3 + 3662q^2 - 2350q + 808 \tag{6.29}$$

$$det(T_{hc,5,1}) = (q-1)^{78}(q-2)^{24} \tag{6.30}$$

$$Tr(T_{hc,5,1}) = 5q^8 - 60q^7 + 336q^6 - 1152q^5 + 2676q^4 - 4374q^3$$

$$+4999q^2 - 3720q + 1429 \tag{6.31}$$

$$det(T_{hc,5,2}) = (q-1)^{66}(q-2)^{16} \tag{6.32}$$

$$Tr((T_{hc,5,2}) = 10q^6 - 96q^5 + 422q^4 - 1084q^3 + 1742q^2 - 1686q + 805 \tag{6.33}$$

$$det(T_{hc,5,3}) = (q-1)^{30}(q-2)^4 \tag{6.34}$$

$$Tr(T_{hc,5,3}) = 10q^4 - 68q^3 + 202q^2 - 302q + 207 \tag{6.35}$$

$$det(T_{hc,5,4}) = (q-1)^6 \tag{6.36}$$

$$Tr(T_{hc,5,4}) = 5q^2 - 18q + 24 \tag{6.37}$$

The matrix $T_{hc,5,4}$ is sufficiently small that we can list it here:

$$T_{hc,5,4} = \begin{pmatrix} D_4 & 2-q & 0 & 0 & 0 & 0 & 0 \\ 3-q & F_{4,3} & 2-q & 1 & 0 & q-3 & 0 \\ 1 & 2-q & F_{4,3} & 3-q & 0 & q-3 & 0 \\ 0 & 0 & 3-q & F_{4,3} & 1-q & 0 & q-3 \\ 0 & 0 & 1 & 2-q & q_{1,2} & 0 & q-3 \\ -1 & q-2 & q-2 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & q-2 & q-1 & 0 & 2 \end{pmatrix} \tag{6.38}$$

Factorizations of characteristic polynomials are given in Table IX.

VII. SOME ALGEBRAIC PROPERTIES OF EVALUATIONS OF TRANSFER MATRICES

We have studied algebraic properties of evaluations of transfer matrices $T_{\Lambda,L_y,d}$ at special values of q. We find many interesting results, and mention only a few here.

First, from our exact results for the $T_{\Lambda,L_y,d}$ for $d = L_y - 1$, we can calculate properties of powers of these matrices, in particular, when evaluated at special values of q. For example,

$$[T_{sq,2,1}]^m = 2^{m-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{for} \quad q = 1, \quad n \ge 1 \tag{7.1}$$

and, for odd $m = 2n+1$ and even $m = 2n$,

$$[T_{sq,2,1}]^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad q = 2, \quad n \ge 0 \tag{7.2}$$

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$$[T_{sq,2,1}]^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad q = 2, \quad n \geq 1 \tag{7.3}$$

and

$$[T_{sq,2,1}]^m = (-1)^m 2^{m-1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{for} \quad q = 3, \quad m \geq 1 . \tag{7.4}$$

Two other identities are given below (where $n \geq 1$ and there is no sum on j):

$$[(T_{sq,4,3})^{2n}]_{jj} = 2^{n-1}(1 + 2^{n-1}) \quad \text{for} \quad q = 3 \tag{7.5}$$

$$[(T_{sq,6,5})^{2n}]_{jj} = \frac{1}{3}\left[2^{2n-1} + 3^n + 1\right] \quad \text{for} \quad q = 3 . \tag{7.6}$$

We next consider transfer matrices for strips of the triangular lattice. In general, from eq. (3.7), it follows that the evaluation of T_{tri,L_y,L_y-1} at $q = 2$ is noninvertible and the evaluation of this matrix at $q = 3$ is noninvertible if $L_y \geq 3$. First, we observe that

$$[T_{tri,2,1}]^m = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad q = 2 . \tag{7.7}$$

The matrix $T_{tri,2,1}$ at $q = 3$ and its powers form the cyclic group of order 3, $\mathbb{Z}_3$, with $[T_{tri,2,1}]^{m+3} = [T_{tri,2,1}]^m$ and

$$T_{tri,2,1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} , \quad [T_{tri,2,1}]^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} , \quad [T_{tri,2,1}]^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{for} \quad q = 3 \tag{7.8}$$

Given that the transfer matrices are real, if such a matrix satisfies $(T_{\Lambda,L_y,d})^p = I_k$, where I_k denotes the $k \times k$ identity matrix, this implies that $det(T_{\Lambda,L_y,d}) = \pm 1$, and if p is odd, as is the case here, with $p = 3$, then $det(T_{\Lambda,L_y,d}) = 1$. Referring to our general result (3.7), we see that this condition is, indeed, met with $k = 2$ if $L_y = 2$, $d = 1$, and $q = 3$.

The matrix $T_{tri,3,0}$ evaluated at $q = 2$ satisfies the relation (denoting this by T for short) $T^3 = T$, so, together with the 2×2 identity matrix I_2, $(T_{tri,3,0})_{q=2}$, $[(T_{tri,3,0})_{q=2}]^2$ form a multiplicative semigroup of order 3. (Here, we recall that a semigroup satisfies the same axioms as a group but without the axiom that inverses of elements must exist; this is also called a semigroup with unit or monoid [67]. The same matrix evaluated at $q = 3$ is idempotent, i.e., satisfies $T^2 = T$, so that I_2 and $(T_{tri,3,0})_{q=3}$ form a semigroup of order 2. One surmises that connections can be made between these cycle lengths and the dependence of the chromatic number of the cyclic strip of the triangular lattice on its length (see appendix), but we have not pursued this.

The matrix $T_{tri,3,1}$ evaluated at $q = 3$ satisfies the relation (again denoting this by T for short) $T^4 = T$ so that I_4, $(T_{tri,3,1})_{q=3}$, $[(T_{tri,3,1})_{q=3}]^2$, and $[(T_{tri,3,1})_{q=3}]^3$ form a semigroup of order 4. The matrix $T_{tri,3,2}$ evaluated at $q = 3$ satisfies the relation $T^3 = T$, and hence the elements I_3, $(T_{tri,3,2})_{q=3}$, and $[(T_{tri,3,2})_{q=3}]^2$ form a semigroup of order 3. (These are not groups since $(T_{tri,3,1})_{q=3}$ and $(T_{tri,3,2})_{q=3}$ are not invertible.) We have found a number of similar algebraic properties of powers of the $T_{\Lambda,L_y,d}$ matrices evaluated for special values of q.

VIII. DISCUSSION OF DETERMINANTS

A. Square-Lattice Strips

Here we discuss further the determinants $det(T_{\Lambda,L_y,d})$ for $\Lambda = sq, tri, hc$, starting with the square lattice . We first consider the zeros of the determinants, restricting to the range $0 \le d \le L_y - 1$ since $T_{sq,L_y,L_y} = (-1)^{L_y}$ has no zeros. From our theorems in Ref. [39], it follows that $det(T_{sq,L_y,L_y-1})$ has (i) the factor $(q-1)$ always; (ii) the factor $(q-3)$ if $L_y = 0$ mod 2; (iii) the factors $(q-2)$ and $(q-4)$ if $L_y = 0$ mod 3; further, all of the zeros of $det(T_{sq,L_y,L_y-1})$ lie in the interval $1 \le q \le 5$ and, as $L_y \to \infty$, they become dense in this interval. One may also investigate the properties of the zeros of $T_{sq,L_y,d}$ for other values of d than $L_y - 1$. Below we list some zeros; for multiple zeros, we indicate the multiplicity in parentheses, $(m = 2)$, etc.

$$det(T_{sq,2,0}) = 0 \quad \text{at} \quad q = \frac{1}{2}(3 \pm i\sqrt{3}\,) = 1.5 \pm 0.866i \tag{8.1}$$

$$det(T_{sq,3,0}) = 0 \quad \text{at} \quad q = 1, \quad 1.659 \pm 1.1615i, \quad 2.682 \tag{8.2}$$

$$det(T_{sq,3,1}) = 0 \quad \text{at} \quad q = 1, \quad 1.229 \pm 0.712i, \quad 2\ (m=2), \quad 3.271 \pm 0.449i \tag{8.3}$$

$$det(T_{sq,3,2}) = 0 \quad \text{at} \quad q = 1,\ 2,\ 4 \tag{8.4}$$

$$det(T_{sq,4,0}) = 0 \quad \text{at} \quad q = 1,\ 1.163 \pm 0.539i,\ 1.355,\ 1.730 \pm 1.331i,\ 2.437, 3, 3.210 \pm 0.6918i \tag{8.5}$$

$$det(T_{sq,4,1}) = 0 \quad \text{at} \quad q = 1\ (m=3),\ 1.257 \pm 0.578i,\ 1.302,\ 1.5 \pm 0.866i\ (m=2),$$

$$1.807 \pm 0.936i, 2.736, \ 3(m = 5), \ 3.4169 \pm 0.6339i \tag{8.6}$$

$$det(T_{sq,4,2}) = 0 \quad \text{at} \quad q = 1 \ (m = 2), \ 1.180 \pm 0.600i, \ 1.327, \ 1.580 \pm 0.606i,$$

$$2.198 \pm 0.573i, \ 3 \ (m = 2), \ 3.839, \ 3.959 \pm 0.294i \tag{8.7}$$

$$det(T_{sq,4,3}) = 0 \quad \text{at} \quad q = 1, \ 1.586, \ 3, \ 4.414 \tag{8.8}$$

and so forth for higher L_y values. We plot these zeros in Figs. 4-5. (The figures are formatted such that, to save space, the vertical axis is shown intersecting the horizontal axis slightly to the left of $q = 1$.)

We observe that all of the zeros of $det(T_{sq,L_y,d})$ that we have calculated satisfy the condition $|q - 3| \leq 2$. As is evident in eq. (8.15), some zeros of $det(T_{hc,L_y,d})$ have real parts that lie outside the interval $1 \leq Re(q) \leq 5$. It would be of interest to prove a theorem bounding the complex zeros of the determinants analogous to results that have been obtained for chromatic zeros, such as the theorem in Ref. [68]. We also note that for $d \neq 0$ (as well as $d \neq L$), for the cases that we have calculated, $det(T_{sq,L_y,d})$ has the factor $(q - 1)$. Although $det(T_{sq,2,0}) = T_{sq,2,0}$ does not have this factor, we observe that for the cases we have calculated with $L_y \geq 3$, $det(T_{sq,L_y,0})$ contains the factor $(q - 1)$.
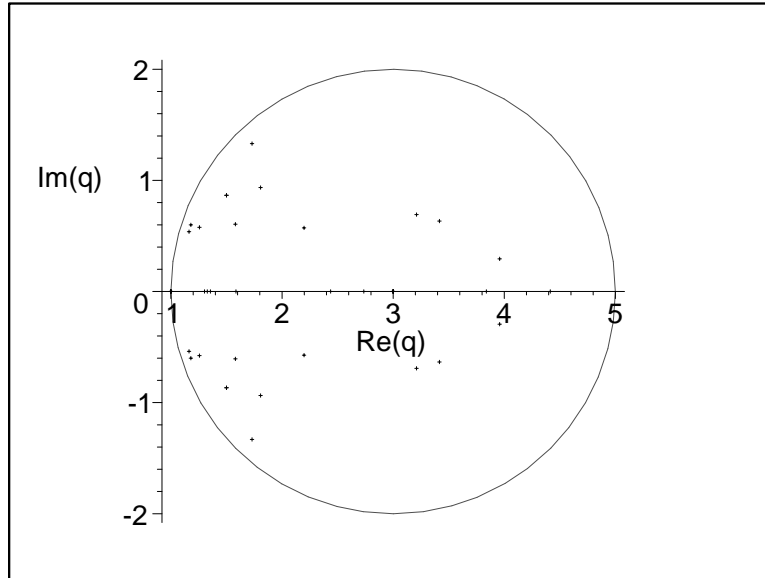

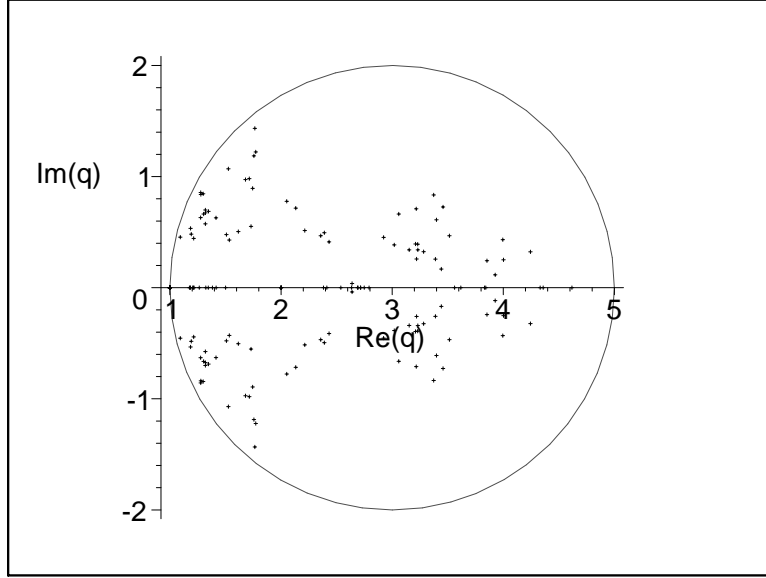FIG. 4. Zeros of $det(T_{sq,L})$ for $L = 4$ in the complex q plane.

FIG. 5. Zeros of $det(T_{sq,L})$ for $L = 5$ in the complex q plane.

B. Triangular-Lattice Strips

An interesting finding is that the determinants of $T_{tri,L_y,d}$ are the simplest among the regular homopolygonal lattices, square, triangular, and honeycomb. Restricting to the range $0 \leq d \leq L_y - 1$ to exclude $T_{tri,L_y,L_y} = (-1)^{L_y}$, which has no zeros, we observe that all of the results that we have obtained are consistent with the following formula, which we conjecture to hold for general L_y (over the range $L_y \geq 2$ where the triangular lattice strip is defined):

$$det(T_{tri,L_y,d}) = \eta(q - 2)^a(q - 3)^b \qquad (8.9)$$

where $\eta = \pm 1$ and

$$a = 2n_P(tri, , L_y - 1, d) + (L_y - 2)n_P(tri, L_y - 2, d) \qquad (8.10)$$

and

$$b = (L_y - 2)n_P(tri, L_y - 1, d) . \qquad (8.11)$$

Here the sign η depends on L_y and d, and we take $n_P(tri, 1, d) \equiv n_P(sq, 1, d)$, in accord with the general equality $n_P(tri, L_y, d) = n_P(sq, L_y, d)$ [9]. (It is not necessary to give a formal definition of $n_P(tri, 0, d)$ because the only time it appears in the equation for a, for

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the lowest nontrivial case $L_y = 2$, it is multiplied by zero.) Some structural properties of this conjecture are indirectly motivated by the exact formula that we have found in Ref. [66] for $det(T_{Z,tri,L_y,d})$, where $T_{Z,tri,L_y,d}$ is the transfer matrix, in the degree-d subspace, for the full Potts model partition function on the given strip. However, aside from the factors of $(v+1)$ in $T_{Z,tri,L_y,d}$ that cause it to vanish in the special case $v = -1$ corresponding to the chromatic polynomial, the rest of the expression that we have established for $det(T_{Z,tri,L_y,d})$ does not involve factors that reduce to powers of $(q-2)$ or $(q-3)$ when $v = -1$; instead, it involves powers of $(q+v)$ and v, which reduce to $(q-1)$ and ± 1. The main structural feature of our known result for $det(T_{Z,tri,L_y,d})$ that influenced our present conjecture is that the powers of these factors involve quantities such as $n_Z(L_y, d)$ and $n_Z(L_y - 1, d)$, where $n_Z(L_y, d)$ is the dimension of $T_{Z,\Lambda,L_y,d}$ (which is the same for $\Lambda = sq, tri, hc$) [9,10]. This led us to construct and test conjectured forms for the chromatic polynomial $det(T_{tri,L_y,d})$ that involve powers of the analogous dimensions $n_P(tri, L_y, d)$.

The conjecture (8.9) implies that the zeros of these determinants $det(T_{tri,L_y,d})$ occur at the two points $q = 3$ and $q = 2$ (for $0 \le d \le L_y - 1$), in accordance with all of our explicit calculations. It is interesting to observe that these are, respectively, the chromatic number $\chi(tri)$ and $\chi(tri) - 1$ of the two-dimensional lattice. For the special case $d = L_y - 1$, eq. (8.9) (with the plus sign) reduces to our previously proved result (3.7). For $L_y \ge 2$ and $d = L_y - 2$, eq. (8.9) can be written explicitly as

$$det(T_{tri,L_y,L_y-2}) = \eta(q-2)^{3L_y-4}(q-3)^{(L_y-1)(L_y-2)} \tag{8.12}$$

Starting from the other end of the range of d, for $d = 0, 1$ (and $L_y \ge 2$), eq. (8.9) can be written explicitly in terms of the Motzkin numbers M_n defined in eq. (2.12) as

$$det(T_{tri,L_y,0}) = \eta(q-2)^{2M_{L_y-2}+(L_y-2)M_{L_y-3}}(q-3)^{(L_y-2)M_{L_y-2}} \tag{8.13}$$

$$det(T_{tri,L_y,1}) = \eta(q-2)^{2M_{L_y-1}+(L_y-2)M_{L_y-2}}(q-3)^{(L_y-2)M_{L_y-1}} \tag{8.14}$$

Similar explicit formulas can be obtained from (8.9) for other values of d.

C. Honeycomb-Lattice Strips

In the case of the honeycomb lattice, we have

$$det(T_{hc,2,0}) = 0 \quad \text{at} \quad 0.691 \pm 0.951i, \ 1.809 \pm 0.588i \tag{8.15}$$

$$det(T_{hc,2,1}) = 0 \quad \text{at} \quad 1 \ (m = 2), \ 2 \pm i \tag{8.16}$$

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As is evident in our results above, for $0 \leq d \leq 2$, the zeros of $det(T_{hc,3,d})$ occur at the two points $q = 1$ and $q = 2$. Restricting to $0 \leq d \leq L_y - 1$ to exclude the constant $T_{hc,L_y,L_y} = 1$, and to odd widths L_y, we observe that all of our results are consistent with the following formula, which we conjecture to hold in general:

$$det(T_{hc,L_y,d}) = (q-1)^s (q-2)^t \quad \text{for odd} \quad L_y \tag{8.17}$$

where

$$s = (L_y + 1)n_P(hc, L_y - 1, d) \tag{8.18}$$

and

$$t = (L_y - 1)n_P(hc, L_y - 2, d) \tag{8.19}$$

where we take $n_P(hc, 1, d) \equiv n_P(sq, 1, d)$. The motivations for this conjecture are based in part on the exact expression that we have found elsewhere [66] for $det(T_{Z,hc,L_y,d})$, where $T_{Z,hc,L_y,d}$ is the transfer matrix, in the degree-d subspace, for the partition function of the full Potts model on the given strip. However, we note that our exact expression for $det(T_{Z,hc,L_y,d})$ does not have a factor that reduces to a power of $(q-2)$ for the special case $v = -1$ that defines the chromatic polynomial. Thus, as in the case of the other lattices, the detailed structure of the determinants of $T_{Z,\Lambda,L_y,d}$ for the full partition function and $T_{P,\Lambda,L_y,d}$, denoted here simply as $T_{\Lambda,L_y,d}$, for the special case $v = -1$ are different. This is not surprising, since the matrix $T_{P,\Lambda,L_y,d}$ does not arise, in general, from a block decomposition of $T_{Z,\Lambda,L_y,d}$ but instead by removing zero columns and corresponding rows of the latter matrix. It is interesting that the zeros of $det(T_{hc,L_y,d})$ occur at $q = \chi(hc)$ and $q = \chi(hc) - 1$, where $\chi(hc) = 2$ is the chromatic number for the honeycomb lattice. For the special case $d = L_y - 1$, eq. (8.17) agrees with the result (3.30) that we have proved from our general calculation of T_{hc,L_y,L_y-1}. For the special case $d = L_y - 2$, eq. (8.17) can be written explicitly as

$$det(T_{hc,L_y,L_y-2}) = (q-1)^{(L_y+1)(3L_y-5)/2}(q-2)^{L_y-1} \quad \text{for odd} \quad L_y . \tag{8.20}$$

Similar explicit formulas can be obtained from (8.17) for other values of d. Both of our conjectured general formulas for $det(T_{\Lambda,L_y,d})$ have the form

$$det(T_{\Lambda,L_y,d}) = \eta \Big[q - (\chi(\Lambda) - 1)\Big]^{p_1} \Big[q - \chi(\Lambda)\Big]^{p_2} \quad \text{for} \quad \Lambda = tri, hc \tag{8.21}$$

where $\chi(\Lambda)$ is the chromatic number of the respective two-dimensional lattices $\Lambda = tri, hc$; $\eta = \pm 1$ for $\Lambda = tri$ and $\eta = 1$ for $\Lambda = hc$; $L_y \geq 3$ is odd for the honeycomb-lattice strips; and the powers p_1 and p_2 have been given in (8.10), (8.11), (8.18), and (8.19).

IX. ACCUMULATION LOCUS OF CHROMATIC ZEROS

Although the locus $\mathcal{B}$ for a strip with a given width L_y and free longitudinal boundary conditions (and free or periodic transverse boundary conditions) is different from that for the corresponding strip with periodic or twisted periodic longitudinal boundary conditions, one notices that in some ways these loci become more similar as L_y increases. One expects, for example, that in the limit $L_y \to \infty$, with any of these boundary conditions, one will obtain the same value of q_c for the $\mathcal{B}$ for the Potts model on the resultant infinite 2D lattice. From our new exact calculations, we find the following results. First, for the $5 \times \infty$ cyclic/Möbius strip of the triangular lattice, $\mathcal{B}$ crosses the real q axis at $q = 0, 2, 3$, and the maximal point

$$q_c = 3.33245.. \ , \quad \Lambda = tri, \quad L_y \times L_x = 5 \times \infty, \quad \mathrm{BC} \ = \ \mathrm{cyclic/Mobius} \ . \tag{9.1}$$

From our new calculations for the $L_y = 4$ and $L_y = 5$ cyclic/Möbius strips of the honeycomb lattice and our analysis of the locus $\mathcal{B}$ in the $L_x \to \infty$ limit, we find that in both cases $\mathcal{B}$ crosses the real q axis at the points $q = 0$ and $q = 2$, and at the maximal crossing points

$$q_c = 2.15476.. \ , \quad \Lambda = hc, \quad L_y \times L_x = 4 \times \infty, \quad \mathrm{BC} \ = \ \mathrm{cyclic/Mobius} \tag{9.2}$$

$$q_c = 2.26407... \ , \quad \Lambda = hc, \quad L_y \times L_x = 5 \times \infty, \quad \mathrm{BC} \ = \ \mathrm{cyclic/Mobius} \ . \tag{9.3}$$

In Table X we present a summary of these new results, together with the results that we have obtained for smaller values of L_y in earlier work. Our present calculations are in agreement with our previous conjecture, based on all of the exact results that we had obtained, that for the infinite-length limit of a cyclic or Möbius strip graph of a given lattice Λ, with a given width L_y, q_c is a nondecreasing function of L_y. For each type of lattice, the values of q_c increase toward the values for the infinite 2D lattices, $q_c(sq) = 3$ [40], $q_c(tri) = 4$ [42], and, formally, $q_c(hc) = (3 + \sqrt{5})/2 \simeq 2.618$ [48,43]. Although the values of q_c for each of these types of lattice strips are less than the value of q_c for the respective two-dimensional lattices, we have shown previously that this need not necessarily be the case. For example, for (the $L_x \to \infty$ limit of) the $L_y = 3$ strip of the square lattice with toroidal boundary conditions [32], $q_c = 3$, equal to the value for the square lattice, and, indeed, for the cyclic self-dual strip of the square lattice, $q_c = 3$ for each of the widths for which we obtained exact calculations [69,11,70]. Nevertheless, the observed monotonically nondecreasing behavior of the q_c values provides a nice interpolation between the value $q_c = 2$ for the one-dimensional Potts antiferromagnet and the values on the respective two-dimensional lattices. As noted before, in general, if one uses strips with free longitudinal boundary conditions, the locus $\mathcal{B}$

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does not necessarily cross the real q axis [23], and in cases where it does not, there is, strictly speaking, no $q_c(\{G\})$. In these cases, the way that one gains information is different; one examines the endpoints of the rightmost arcs on $\mathcal{B}$ that are closest to the real axis. These are found to exhibit the general tendency to move gradually to the right with endpoints that move closer to the real axis as the strip width is increased [23,24,34,57–59,35,39,36]. The results are consistent with the inference that as $L_y \to \infty$, the arcs will merge to form closed curves on $\mathcal{B}$ and the right-most part of $\mathcal{B}$ will cross the real axis at the value of q_c for the corresponding two-dimensional lattice. (One can attempt to carry out a similar type of study for three-dimensional lattices, and exploratory work has been done in [38].)

In Figs. 6-9 we show the singular loci $\mathcal{B}$ for infinite-length strips of the triangular lattice with width $L_y = 5$ and of the honeycomb lattice with widths $L_y = 3, 4, 5$, together with zeros of the chromatic polynomials (chromatic zeros) for long finite-length strips of each type. We first discuss the triangular-lattice strip (Fig. 6). This should be compared with our earlier results for the $L_y = 2$ strip in Fig. 3 in Ref. [30] and for the $L_y = 3$ and $L_y = 4$ strips in Figs. 2 and 3 of Ref. [36]. As expected, the comparison is particularly close with the $L_y = 4$ case. For $L_y = 3, 4$ we found that there are two inner curves that run between points on the outer envelope and cross the real q axis at $q = 2$ and $q = 3$, separating the interior of the envelope curve into regions that include the real segments (i) $0 \leq q \leq 2$, (ii) $2 \leq q \leq 3$, and (iii) $3 \leq q \leq q_c$. We also found that the densities of zeros on these curves passing through $q = 2$ and $q = 3$ were slightly less than the densities of the zeros on the outer envelope curves. Another feature that we found was the existence of two "bubble" regions protruding to the right from the main curve that passes through q_c. Finally, we observed that as L_y increases from $L_y = 3$ to $L_y = 4$, there are more zeros, and more support for a part of $\mathcal{B}$ in the half-plane with $Re(q) < 0$. Our current results for $L_y = 5$ exhibit these same features, and show an increase, relative to our $L_y = 4$ results, in the portion of $\mathcal{B}$ extending into the half-plane with $Re(q) < 0$. This is indicated in Table X. In this table we also characterize the form of the curve $\mathcal{B}$ as it passes through the point $q = 0$; given that it passes vertically through this point, and is invariant under complex conjugation, there are two possibilities, viz., that it extends into the half-plane with (i) $Re(q) > 0$ or (ii) $Re(q) < 0$. We denote these two possibilities as (i) convex and (ii) concave to the left. Note that, *a priori*, the conditions that $\mathcal{B}$ has support in the half-plane with $Re(q) < 0$ and that it is concave to the left are not equivalent since it might be convex to the left, i.e., curve into the $Re(q) > 0$ half-plane in the neighborhood of the origin, but then curve back to the left at larger values of $|Im(q)|$ and cross over into the $Re(q) < 0$ half-plane. However, we find that this does not happen for any of the strips for which we have done calculations, as is evident in Table X. Thus, for these strips, the properties that $\mathcal{B}$ has support for $Re(q) < 0$ and that it is concave to the

left at $q = 0$ are observed to occur together. We also note that the zeros on the right-hand side of the locus $\mathcal{B}$ cross the real axis at a point consistent with the asymptotic result for q_c given in eq. (9.1). We observe small complex-conjugate bubble regions around $q = 2.6 \pm 2i$, where the curve passing through $q = 2$ intersects the outer curve on $\mathcal{B}$. We should also remark on the parts of the locus $\mathcal{B}$ near the points $q = 1 \pm 2.1i$: without magnification, these appear to be arc endpoints, but in fact the end in small, very narrow, bubble regions. This is consistent with our finding for the $L_x \to \infty$ limit of all strips of regular lattices with periodic longitudinal boundary conditions, that the respective loci $\mathcal{B}$ do not have arc endpoints, in contrast with the situation for strips with free longitudinal boundary conditions, for which the loci $\mathcal{B}$ generically do exhibit such arc endpoints. In our earlier work we have found that loci $\mathcal{B}$ can exhibit tiny sliver regions, and we mention that for this case of the $L_y = 5$ cyclic strip of the triangular lattice and the for the loci shown below for the strips of the honeycomb lattice, such tiny sliver regions, if small enough, would elude our analysis because of the finite size of the grid that we use for testing for equimodular dominant λ's.

In addition to the continuous accumulation set of chromatic zeros that forms $\mathcal{B}$, one may also investigate discrete chromatic zeros. In general, for any graph with at least one edge, there is always a chromatic zero at $q = 1$, since there is no proper coloring of such a graph with just one color. For strips of the triangular lattice with width the $L_y = 5$ and reasonably great length, we also find a chromatic zero near to the point $q = (3 + \sqrt{5})/2 \simeq 2.618$, which is a zero of $c^{(2)}$, where a degree $d = 2$ λ is dominant, and a chromatic zero near to the point $q \simeq 3.2470$, a zero of $c^{(3)}$, where a degree $d = 3$ λ is dominant.

We next show the singular loci $\mathcal{B}$ for the infinite-length strips of the honeycomb lattice with $L_y = 3, 4, 5$ in Figs. 7-9, together with chromatic zeros for long finite-length strips of each respective width. As background, we recall that in the $L_x \to \infty$ limit for the smaller widths $L_y = 2$ (Ref. [27]) and $L_y = 3$ (Ref. [10]), we found that six curves on $\mathcal{B}$, forming three branches, intersect at the point q_c (which is equal to 2 for those widths). Hence, it was found that the region diagram included at least six regions (which comprised the totality of regions for $L_y = 2$ and were augmented by two additional very small regions centered approximately at $q = 0.5 \pm 0.45i$ for $L_y = 3$): (i) the outermost region R_1, extending infinitely far away from the origin and including the intervals $q > 2$ and $q < 0$ on the real axis; (ii) the innermost region R_2, which includes the interval $0 \leq q \leq 2$; (iii) a complex-conjugate (c.c.) pair of regions R_3, R_3^* forming upper and lower outer crescent-shaped areas adjacent to q_c, and (iv) the c.c. pair R_4, R_4^* forming upper and lower inner crescent-shaped areas adjacent to q_c. It was also found that the density of zeros on the inner two of the six curves passing through q_c was somewhat lower than the density of the zeros on other parts of $\mathcal{B}$. The locus $\mathcal{B}$ in the vicinity of the origin $q = 0$ is concave to the left, which implies that it has support for values

of q in the half-plane with $Re(q) < 0$. This concavity increases as the width increases. In the loci $\mathcal{B}$ presented in Fig. 8 and 9, we see that the point $q = 2$ is an intersection point of four rather than six curves, and this feature is correlated with the fact that for these widths, q_c exceeds 2. Thus, in the plot for $L_y = 4$ we see a small self-conjugate bubble region that includes the interval $2 \leq q \lesssim q_{hc4c}$, where $q_{hc4c} \simeq 2.155$ is the value of q_c for this strip, given in eq. (9.2). For $L_y = 5$ we observe three small self-conjugate regions in $\mathcal{B}$ (see Fig. 9) that include the three segments $2 \leq q \leq q_{hc5a}$, $q_{hc5a} \leq q \leq q_{hc5b}$, and $q_{hc5b} \leq q \leq q_{hc5c}$, where $q_{hc5a} \simeq 2.1997$, $q_{hc5b} \simeq 2.2468$, and $q_{hc5c} \simeq 2.26407$ is the value of q_c for this strip, given in eq. (9.3). Thus, the locus $\mathcal{B}$ for the cyclic $L_y = 5$ strip of the honeycomb lattice crosses the real q axis $q = 0$, 2, q_{hc5a}, q_{hc5b}, and q_{hc5c}. These values are listed, together with those for other strips, in Table X. For the $L_y = 3$ strip, one sees in Fig. 7 a complex-conjugate pair of narrow regions around $q \simeq 0.5 \pm 1.4i$; similarly, for the $L_y = 4$ strip, one sees in Fig. 8 a complex-conjugate pair of very narrow sliver regions around $q \simeq 0.1 \pm 1.3i$. For the $L_y = 5$ strip of the honeycomb lattice, we find arcs ending in tiny bubble regions at approximately $q = -0.1 \pm i$. A general feature that we observe is increasing complexity of the loci $\mathcal{B}$ with increasing strip width. As before, we note that regions that are extremely small could be missed by our grid used in testing for degeneracies between dominant eigenvalues that yield components of the locus $\mathcal{B}$.

FIG. 6. Singular locus $\mathcal{B}$ in the complex q plane for the $L_x \to \infty$ limit of the cyclic strip of the triangular lattice with width $L_y = 5$. For comparison, the plot also shows zeros of $P(tri, L_y \times L_x, cyc., q)$ for $L_y = 5$ and a typical large value of the length, $L_x = 20$.
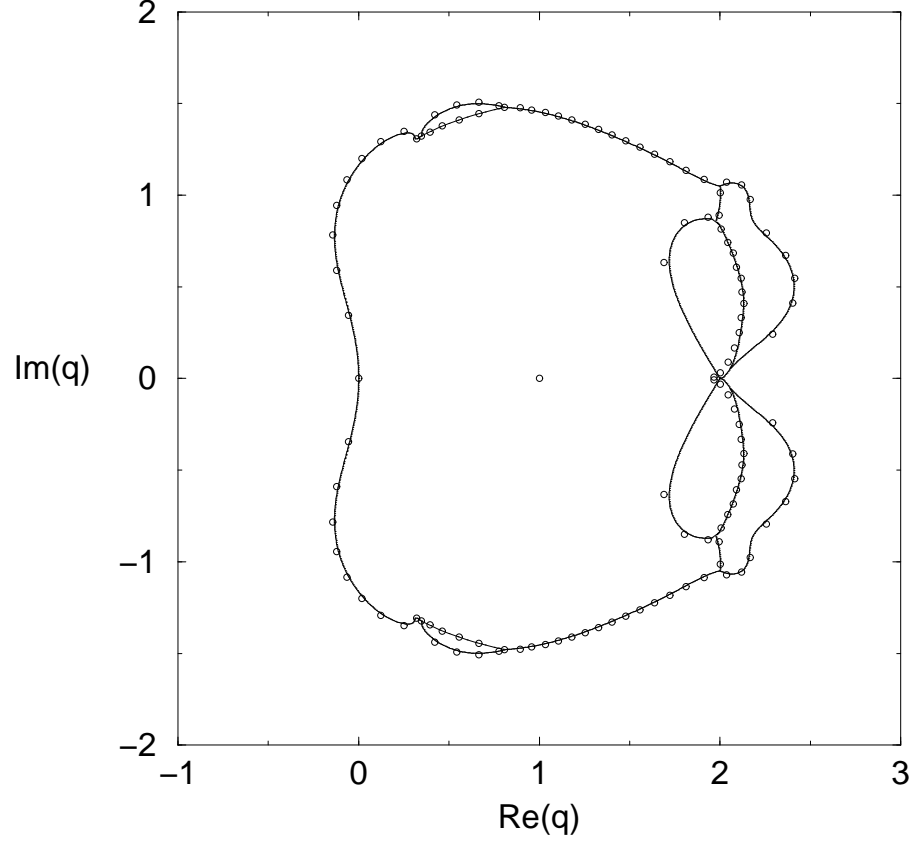
FIG. 7. Singular locus $\mathcal{B}$ in the complex q plane for the $L_x \to \infty$ limit of the cyclic strip of the honeycomb lattice with width $L_y = 3$. For comparison, the plot also shows zeros of $P(tri, L_y \times L_x, cyc., q)$ with $L_y = 3$ and a typical large value of the length, $L_x = 40$.
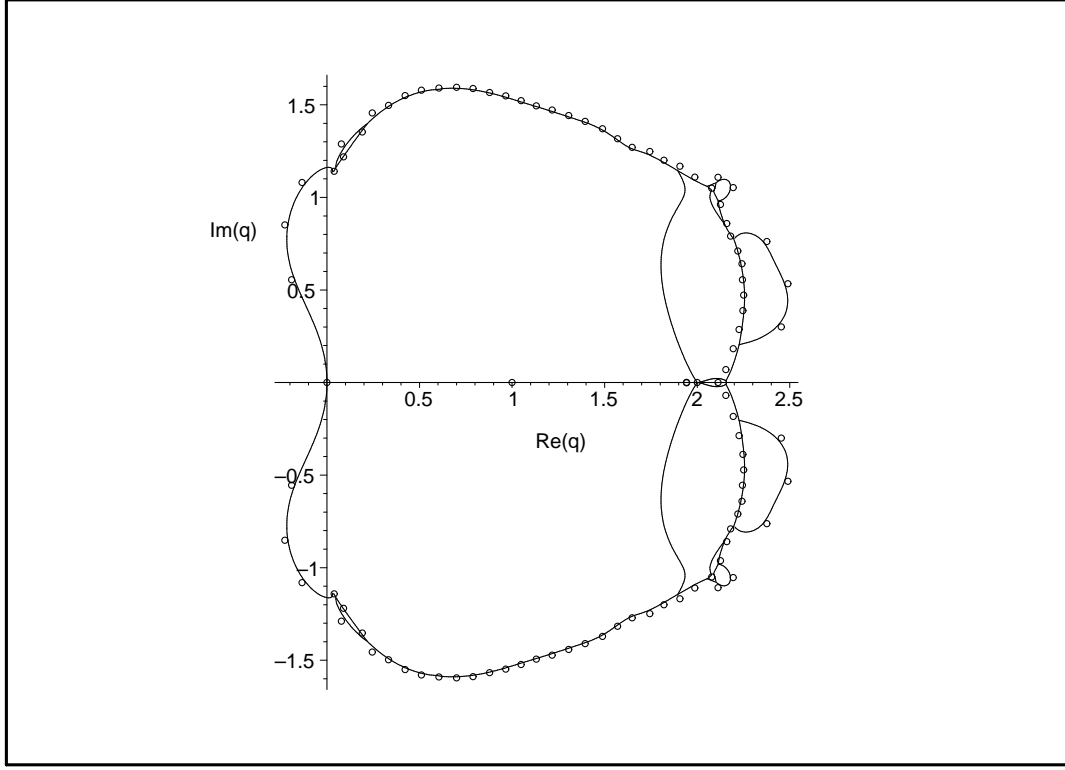
FIG. 8. Singular locus $\mathcal{B}$ in the complex q plane for the $L_x \to \infty$ limit of the cyclic strip of the honeycomb lattice with width $L_y = 4$. For comparison, the plot also shows zeros of $P(tri, L_y \times L_x, cyc., q)$ with $L_y = 4$ and a typical large value of the length, $L_x = 24$.
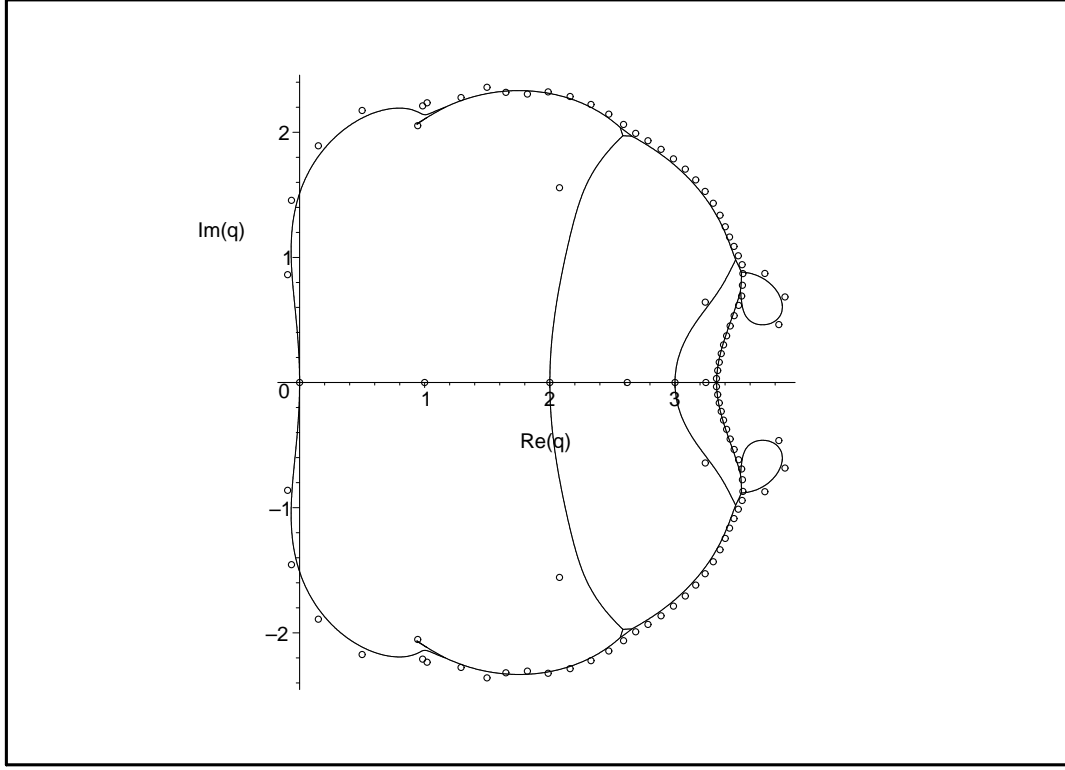
51
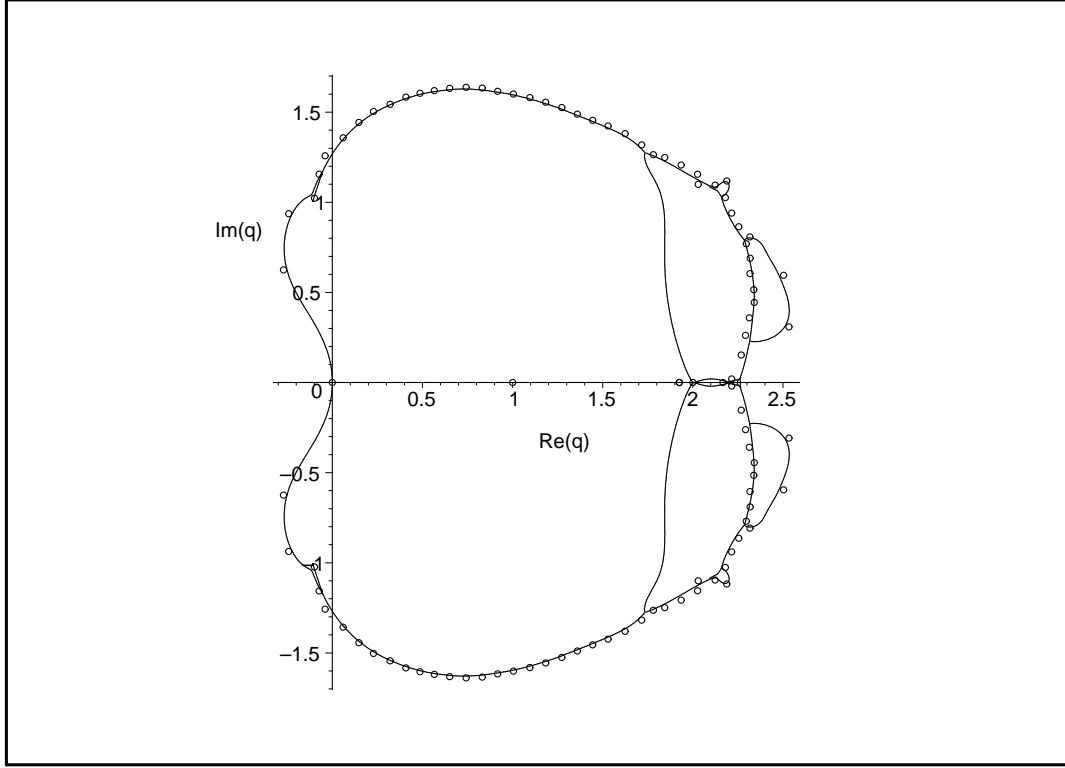
FIG. 9. Singular locus $\mathcal{B}$ in the complex q plane for the $L_x \to \infty$ limit of the cyclic strip of the honeycomb lattice with width $L_y = 5$. For comparison, the plot also shows zeros of $P(tri, L_y \times L_x, cyc., q)$ with $L_y = 5$ and a typical large value of the length, $L_x = 20$.

X. APPENDIX 1: CHROMATIC NUMBERS FOR LATTICE STRIPS

The following chromatic numbers apply for lengths that are greater than the lowest few where strips sometimes degenerate. For cyclic strips of the square lattice, the chromatic number is given by

$$\chi(sq, L_y \times L_x, cyc.) = \begin{cases} 2 & \text{for even } L_x \\ 3 & \text{for odd } L_x \end{cases} \tag{10.1}$$

independent of L_y. For the Möbius strips of the square lattice (e.g., [35])

$$\chi(sq, L_y \times L_x, Mb.) = \begin{cases} 2 & \text{for } (L_x, L_y) = (e, o), \ (o, e) \\ 3 & \text{for } (L_x, L_y) = (e, e), \ (o, o) \end{cases} \tag{10.2}$$

where e and o denote even and odd.

The cyclic and Möbius strips of the triangular lattice have chromatic numbers (e.g. [36])

$$\chi(tri, L_y \times L_x, cyc.) = \begin{cases} 3 & \text{if } L_x = 0 \mod 3 \\ 4 & \text{if } L_x = 1 \ \text{ or } L_x = 2 \mod 3 \end{cases} \tag{10.3}$$

$$\chi(tri, L_y \times L_x, Mb.) = 4 \tag{10.4}$$

Again for L_x above the first few values ($L_x = 1, 2$ where strips can degenerate), the cyclic and Möbius strips of the honeycomb lattice have chromatic numbers (e.g. [10])

$$\chi(hc, L_y \times L_x, cyc.) = 2 \tag{10.5}$$

$$\chi(hc, L_y \times L_x, Mb.) = 3 \tag{10.6}$$

XI. APPENDIX 2: TABLES

TABLE I. Table of numbers $n_P(\Lambda, L_y, d)$ and their sums, $N_{P,\Lambda,L_y,\lambda}$ for cyclic strips of the square and triangular lattices. Blank entries are zero.

$L_y \downarrow \quad d \rightarrow$	0	1	2	3	4	5	6	$N_{P,\Lambda,L_y,\lambda}$
1	1	1						2
2	1	2	1					4
3	2	4	3	1				10
4	4	9	8	4	1			26
5	9	21	21	13	5	1		70

TABLE II. Table of numbers $n_P(hc, L_y, d)$ and their sums, $N_{P,hc,L_y,\lambda}$ for cyclic strips of the honeycomb lattice. Blank entries are zero.

$L_y \downarrow \quad d \rightarrow$	0	1	2	3	4	5	6	$N_{P,hc,L_y,\lambda}$
2	1	2	1					4
3	3	6	4	1				14
4	6	13	11	5	1			36
5	19	43	40	22	7	1		132

TABLE III. Table of $\Delta n_P(sq, L_y, d)$ for strips of the square lattice. Blank entries are zero. The last entry for each value of L_y is the total number of partitions with self-reflection symmetry.

$L_y \downarrow \quad d \rightarrow$	0	1	2	3	4	5	6	7	8	9	10	$\Delta N_{P,L_y}$
1	1	1										2
2	1	0	1									2
3	2	2	1	1								6
4	2	1	2	0	1							6
5	5	5	3	3	1	1						18
6	5	3	5	1	3	0	1					18
7	13	13	9	9	4	4	1	1				54
8	13	9	13	4	9	1	4	0	1			54
9	35	35	26	26	14	14	5	5	1	1		162
10	35	26	35	14	26	5	14	1	5	0	1	162

TABLE IV. Table of numbers $n_P(sq, L_y, d, \pm)$ for strips of the square lattice. For each L_y value, the entries in the first and second lines are $n_P(sq, L_y, d, +)$ and $n_P(sq, L_y, d, -)$, respectively. Blank entries are zero. The last entry for each value of L_y is the total $N_{P,L_y,\lambda}$.

L_y $(d,+)$ $(d,-)$	$0,+$ $0,-$	$1,+$ $1,-$	$2,+$ $2,-$	$3,+$ $3,-$	$4,+$ $4,-$	$5,+$ $5,-$	$6,+$ $6,-$	$7,+$ $7,-$	$8,+$ $8,-$	$9,+$ $9,-$	$10,+$ $10,-$	$N_{P,L_y,\lambda}$
2	1	1	1									
		1										4
3	2	3	2	1								
	1	1										10
4	3	5	5	2	1							
	1	4	3	2								26
5	7	13	12	8	3	1						
	2	8	9	5	2							70
6	13	27	30	20	11	3	1					
	8	24	25	19	8	3						192
7	32	70	77	61	34	15	4	1				
	19	57	68	52	30	11	3					534
8	70	166	199	163	106	49	19	4	1			
	57	157	186	159	97	48	15	4				1500
9	179	435	528	468	318	174	72	24	5	1		
	144	400	502	442	304	160	67	19	4			4246
10	435	1107	1405	1288	946	550	265	96	29	5	1	
	400	1081	1307	1274	920	545	251	95	24	5		12092

TABLE V. Table of $\Delta n_P(hc, L_y, d)$ for strips of the honeycomb lattice with even L_y. Blank entries are zero. The last entry for each value of L_y is the total number of partitions with self-reflection symmetry.

L_y $\downarrow$ $d \to$	0	1	2	3	4	5	6	7	8	9	10	$\Delta N_{P,hc,L_y}$
2	1	0	1									2
4	4	3	3	1	1							12
6	7	4	7	1	4	0	1					24
8	36	30	30	17	17	6	6	1	1			144
10	66	47	66	23	47	7	23	1	7	0	1	288

TABLE VI. Table of numbers $n_P(hc, L_y, d, \pm)$ for strips of the honeycomb lattice with even L_y. For each L_y value, the entries in the first and second lines are $n_P(hc, L_y, d, +)$ and $n_P(hc, L_y, d, -)$, respectively. Blank entries are zero. The last entry for each value of L_y is the total $N_{P,hc,L_y,\lambda}$.

L_y $(d,+)$ $(d,-)$	$0,+$ $0,-$	$1,+$ $1,-$	$2,+$ $2,-$	$3,+$ $3,-$	$4,+$ $4,-$	$5,+$ $5,-$	$6,+$ $6,-$	$7,+$ $7,-$	$8,+$ $8,-$	$9,+$ $9,-$	$10,+$ $10,-$	$N_{P,hc,L_y,\lambda}$
2	1	1	1									
	1											4
4	5	8	7	3	1							
	1	5	4	2								36
6	25	53	56	35	17	4	1					
	18	49	49	34	13	4						358
8	194	454	518	404	237	100	32	6	1			
	158	424	488	387	220	94	26	5				3748
10	1590	4036	4953	4324	2947	1565	664	208	51	7	1	
	1524	3989	4887	4301	2900	1558	641	207	44	7		40404

TABLE VII. Factorization properties of characteristic polynomials $CP(T_{sq,L_y,d}, z)$ for cyclic strips of the square lattices. The notation (m, n) means that $CP(T_{sq,L_y,d}, z)$ factorizes into polynomials in z of degree m and n with integer coefficients. The notation j^k indicates that there are k factors of degree j.

$L_y \downarrow$ $d \rightarrow$	0	1	2	3	4	5	$N_{P,sq,L_y,\lambda}$
1	(1)	(1)					2
2	(1)	(1^2)	(1)				4
3	(2)	(1,3)	(1^3)	(1)			10
4	(1,3)	(4,5)	(3,5)	$(1^2,2)$	(1)		26
5	(2,7)	(8,13)	(9,12)	(5,8)	(2,3)	(1)	70

TABLE VIII. Factorization properties of characteristic polynomials $CP(T_{tri,L_y,d},z)$ for cyclic strips of the triangular lattices. The notation (m,n) means that $CP(T_{tri,L_y,d},z)$ factorizes into polynomials in z of degree m and n with integer coefficients. The notation j^k indicates that there are k factors of degree j.

$L_y \downarrow \quad d \rightarrow$	0	1	2	3	4	5	$N_{P,tri,L_y,\lambda}$
1	(1)	(1)					2
2	(1)	(2)	(1)				4
3	(2)	(1,3)	(1,2)	(1)			10
4	(4)	(9)	(8)	(4)	(1)		26
5	(2,7)	(8,13)	(9,12)	(5,8)	(2,3)	(1)	70

TABLE IX. Factorization properties of characteristic polynomials $CP(T_{hc,L_y,d},z)$ for cyclic strips of the honeycomb lattices. The notation (m,n) means that the characteristic polynomial factorizes into polynomials in z of degree m and n with integer coefficients. The notation j^k indicates that there are k factors of degree j.

$L_y \downarrow \quad d \rightarrow$	0	1	2	3	4	5	$N_{P,hc,L_y,\lambda}$
2	(1)	(1^2)	(1)				4
3	(3)	(6)	(1,3)	(1)			14
4	(1,5)	(5,8)	(4,7)	$(1^2,3)$	(1)		36
5	(19)	(43)	(40)	(22)	(1,6)	1	132

TABLE X. Properties of the locus $\mathcal{B}$ for infinite-length cyclic/Möbius strips with width L_y of regular lattices Λ, including square (sq), triangular (tri), and honeycomb (hc) lattices. The notation BCR denotes $\mathcal{B}$ crossings on the real q axis, the greatest of which is q_c. The notation SN indicates whether $\mathcal{B}$ has some support for values of q with $Re(q) < 0$, marked as y (yes) or n (no). The behavior of $\mathcal{B}$ in the neighborhood of the origin is indicated as convex or concave to the left, with the convention taken such that, for example, the circle $|q - 1| = 1$ is convex (to the left) at $q = 0$. The final column gives the reference.

Λ	L_y	BCR	SN	$\mathcal{B}$ at $q = 0$	Ref.
sq	1	2, 0	n	convex	−
sq	2	2, 0	n	convex	[20]
sq	3	2.34, 2, 0	y	concave	[29]
sq	4	2.49, 2, 0	y	concave	[35]
sq	5	2.58, 2, 0	y	concave	[39]
tri	2	3, 2, 0	n	convex	[30]
tri	3	3, 2, 0	n	convex	[36]
tri	4	3.23, 3, 2, 0	y	concave	[36]
tri	5	3.33, 3, 2, 0	y	concave	here
hc	2	2, 0	y	concave	[27]
hc	3	2, 0	y	concave	[10]
hc	4	2.155, 2, 0	y	concave	here
hc	5	2.26, 2.25, 2.20, 2, 0	y	concave	here

XII. APPENDIX 3

Here we list polynomials that occur in our explicit expressions for elements of transfer matrices. It is convenient to define

$$F_{m,n} = D_m - D_n \tag{12.1}$$

One then sees that $q^2 - 4q + 5$ is $F_{4,3}$. Note that

$$F_{m+2,m} = (q-1)^{m-1}(q-2) \tag{12.2}$$

$$F_{m+4,m} = (q-1)^{m-1}(q-2)(q^2 - 2q + 2) \tag{12.3}$$

$$F_{m+6,m} = (q-1)^{m-1}(q-2)(q^2 - q + 1)D_4$$

$$= (q-1)^{m-1}(q-2)(q^2 - q + 1)(q^2 - 3q + 3) \tag{12.4}$$

etc., so e.g., $F_{4,2} = (q-1)(q-2)$. One can also define

$$G_{m,n} = D_m - 2D_n \tag{12.5}$$

so that another frequently occurring entry, $q^2 - 5q + 7$ is seen to be $G_{4,3}$. We have not tried to carry out such constructions in an exhaustive manner and often just give shorthand notation to commonly occurring polynomials. Because of the various equivalences, there are also several ways of writing a given polynomial.

We also list a set of polynomials that recur frequently and are given shorthand names, p_n, r_n, and s_n for quadratic, cubic, and quartic terms:

$$p_4 = q^2 - 3q + 4, \quad p_6 = q^2 - 4q + 6, \quad p_8 = q^2 - 5q + 8 \tag{12.6}$$

$$p_{10} = q^2 - 6q + 10, \quad p_{13} = q^2 - 7q + 13, \quad p_{14} = q^2 - 7q + 14 \tag{12.7}$$

$$p_{15} = q^2 - 7q + 15, \quad p_{17} = q^2 - 8q + 17 \tag{12.8}$$

$$r_{11} = q^3 - 5q^2 + 11q - 11, \quad r_{13} = q^3 - 6q^2 + 14q - 13 \tag{12.9}$$

$$r_{19} = q^3 - 7q^2 + 19q - 19, \quad r_{20} = q^3 - 7q^2 + 19q - 20 \tag{12.10}$$

$$r_{34} = q^3 - 9q^2 + 29q - 34, \quad r_{46} = q^3 - 11q^2 + 39q - 46 \qquad (12.11)$$

$$r_{47} = q^3 - 10q^2 + 36q - 47, \quad r_{49} = q^3 - 10q^2 + 37q - 49, r_{58} = q^3 - 11q^2 + 43q - 58$$
$$(12.12)$$

$$s_7 = q^4 - 5q^3 + 11q^2 - 12q + 7, \quad s_{21} = q^4 - 7q^3 + 21q^2 - 32q + 21 \qquad (12.13)$$

$$s_{31} = q^4 - 8q^3 + 27q^2 - 45q + 31, s_{47} = q^4 - 9q^3 + 34q^2 - 63q + 47 \qquad (12.14)$$

$$s_{61} = q^4 - 9q^3 + 35q^2 - 70q + 61, s_{82} = q^4 - 11q^3 + 48q^2 - 99q + 82 \qquad (12.15)$$

$$s_{148} = q^4 - 13q^3 + 67q^2 - 160q + 148, s_{173} = q^4 - 13q^3 + 68q^2 - 171q + 173 \qquad (12.16)$$

$$s_{3,5} = (q^2 - 2q + 3)(q^2 - 4q + 5), \quad t_{5,5} = (q^2 - 4q + 5)(q^3 - 4q^2 + 6q - 5) \qquad (12.17)$$

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